Post-Newtonian Gravitational Radiation

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I. INTRODUCTION

A. On approximation methods in general relativity

Let us declare that the most important *devoir* of any physical theory is to draw firm predictions for the outcome of laboratory experiments and astronomical observations. Unfortunately, the devoir is quite difficult to fulfill in the case of general relativity, essentially because of the complexity of the Einstein field equations, to which only few exact solutions are known. For instance, it is impossible to settle the exact prediction of this theory when there are no symmetry in the problem (as is the case in the problem of the gravitational dynamics of separated bodies). Therefore, one is often obliged, in general relativity, to resort to approximation methods.

It is beyond question that approximation methods do work in general relativity. Some of the great successes of this theory were in fact obtained using approximation methods. We have particularly in mind the test by Taylor and collaborators [1–3] regarding the orbital decay of the binary pulsar PSR 1913+16, which is in agreement to within 0.35% with the general-relativistic *post-Newtonian* prediction. However, a generic problem with approximation methods (especially in general relativity) is that it is non trivial to define a clear framework within which the approximation method is mathematically well-defined, and such that the results of successive approximations could be considered as *theorems* following some precise (physical and/or technical) assumptions. Even more difficult is the problem of the relation between the approximation method and the *exact* theory. In this context one can ask: What is the mathematical nature of the approximation series (convergent, asymptotic, ...)? What its "reliability" is (i.e., does the approximation series come from the Taylor expansion of a family of exact solutions)? Does the approximate solution satisfy some "exact" boundary conditions (for instance the no-incoming radiation condition)?

Since the problem of theoretical prediction in general relativity is complex, let us distinguish several approaches (and ways of thinking) to it, and illustrate them with the example of the prediction for the binary pulsar. First we may consider what could be called the "physical" approach, in which one analyses the relative importance of each physical phenomena at work by using crude numerical estimates, and where one uses only the lowest-order approximation, relating if necessary the local physical quantities to observables by means of balance equations (perhaps not well defined in terms of basic theoretical concepts). The physical approach to the problem of the binary pulsar is well illustrated by Thorne in his beautiful Les Houches review [4] (see also the round table discussion moderated by Ashtekar [5]): one derives the loss of energy by gravitational radiation from the (Newtonian) quadrupole formula applied formally to point-particles, assumed to be test-masses though they are really self-gravitating, and one argues "physically" that the effect comes from the variation of the Newtonian binding energy in the center-of-mass frame – indeed, on physical grounds, what else could this be (since we expect the rest masses won't vary)? The physical approach yields the correct result for the rate of decrease of the period of the binary pulsar. Of course, thinking physically is extremely useful, and indispensable in a preliminary stage, but certainly it should be completed by a solid study of the connection to the mathematical structure of the theory. Such a study would *a posteriori* demote the physical approach to the status of "heuristic" approach. On the other hand, the physical approach may fall short in some situations requiring a sophisticated mathematical modelling (like in the problem of the dynamics of singularities), where one is often obliged to follow one's mathematical rather than physical insight.

A second approach, that we shall qualify as "rigorous", has been advocated mainly by Jürgen Ehlers (see, e.g., [6]). It consists of looking for a high level of mathematical rigor, within the exact theory if possible, and otherwise using an approximation scheme that we shall be able to relate to the exact theory. This does not mean that we will be so much wrapped up by mathematical rigor as to forget about physics. Simply, in the rigorous approach, the prediction for the outcome of an experiment should follow mathematically from first theoretical principles. Clearly this approach is the one we should ideally adhere to. As an example, within the rigorous approach, one was not permitted, by the end of the seventies, to apply the standard quadrupole formula to the binary pulsar. Indeed, as pointed out by Ehlers *et al* [7], it was not clear that gravitational radiation reaction on a self-gravitating system implies the standard quadrupole formula for the energy flux, notably because computing the radiation reaction demands *a priori* three non-linear iterations of the field equations [8], which were not fully available at that time. Ehlers and collaborators [7] remarked also that the exact results concerning the structure of the field at infinity (notably the asymptotic shear of null geodesics whose variation determines the flux of radiation) were not connected to the actual dynamics of the binary.

Maybe the most notable result of the rigorous approach concerns the relation between the exact theory and the approximation methods. In the case of the post-Newtonian approximation (limit $c \to \infty$), Jürgen Ehlers has provided with his frame theory [9–11] a conceptual framework in which the post-Newtonian approximation can be clearly formulated (among other purposes). This theory unifies the theories of Newton and Einstein into a single generally covariant theory, with a parameter $1/c$ taking the value zero in the case of Newton and being the inverse of the speed of light in the case of Einstein. Within the frame theory not only does one understand the limit relation of Einstein's theory to Newton's, but one explains why it is legitimate when describing the predictions of general relativity to use the common-sense language of Newton (for instance thinking that the trajectories of particles in an appropriately defined coordinate system take place in some Euclidean space, and viewing the coordinate velocities as being defined with respect to absolute time). It was shown by Lottermoser [12] that the constraint equations of the (Hamiltonian formulation of the) Ehlers frame theory admit solutions with a well-defined post-Newtonian limit. Further in the spirit of the rigorous approach, we quote the work of Rendall [13] on the definition of the post-Newtonian approximation, and the link to the post-Newtonian equations used in practical computations. (See also [14,15] for an attempt at showing, using restrictive assumptions, that the post-Newtonian series is asymptotic.)

The important remarks of Jürgen Ehlers *et al* [7] on the applicability of the quadrupole formula to the binary pulsar stimulated research to settle down this question with (al least) acceptable mathematical rigor. The question was finally answered positively by Damour and collaborators [16–19], who obtained in algebraically closed form the general-relativistic equations of motion of two compact objects, up to the requisite 5/2 post-Newtonian order (2.5PN order or $1/c^5$) where the gravitational radiation reaction force appears. This extended to 2.5PN order the work at 1PN of Lorentz and Droste [20], and Einstein, Infeld and Hoffmann [21]. The net result is that the dynamics of the binary pulsar as predicted by (post-Newtonian) general relativity is in full agreement both with the prediction of the quadrupole formula, as derived earlier within the "physical" approach, and with the observations by Taylor *et al* (see [22] for discussion).

Motivated by the success of the theoretical prediction in the case of the binary pulsar [16–19,22], we shall try to follow in this article the spirit of the "rigorous" approach of Jürgen Ehlers, notably in the way it emphasizes the mathematical proof, but we shall also differ from it by a systematic use of approximation methods. This slightly different approach recognizes from the start that in certain difficult problems, it is impossible to derive a physical result all the way through the exact theory without any gap, so that one must proceed with approximations. *But*, in this approach, one implements a mathematically well-defined framework for the approximation method, and within this framework one proves theorems that (ideally) guarantee the correctness of the theoretical prediction to be compared with experiments. Because the comparison with experiments is the only thing which matters *in fine* for a pragmatist, we qualify this third approach as "pragmatic".

In this article we describe the pragmatic approach to the problem of gravitational radiation emitted by a general isolated source, based on the rigorous post-Minkowskian iteration of the field outside the source [23], and on the general connection of the exterior field to the field inside a slowly-moving source [24,25]. Note that for this particular problem the pragmatic approach is akin to the rigorous one in that it permits to establish some results on the connection between approximate and exact methods. For instance it was proved by Damour and Schmidt [26] (see also [27,28]) that the post-Minkowskian algorithm generates an asymptotic approximation to exact solutions, and it was shown [29] that the solution satisfies to any order in the post-Minkowskian expansion a rigorous definition of asymptotic flatness at future null infinity. However it remains a challenge to analyse in the manner of the rigorous approach the relation to exact theory of the whole formalism of [23–25,29].

By combining the latter post-Minkowskian approximation and a post-Newtonian expansion inside the system, it was proved (within this framework of approximations) that the quadrupole formula for slowly-moving, weakly-stressed and self-gravitating systems is correct, even including post-Newtonian corrections [30]; and *idem* for the radiation reaction forces acting locally inside the system, and for the associated balance equations [31,32]. These results answered positively Ehlers' remarks [7] in the case of slowly-moving extended (fluid) systems. However we are also interested in this article to the application to binary systems of compact objects modelled by point-masses. Indeed the latter sources of radiation are likely to be detected by future gravitational-wave experiments, and thus concern the pragmatist. We shall see how one can address the problem in this case. (When specialized to point-mass binaries, the results on radiation reaction [31,32] are in agreement with separate work of Iyer and Will [33,34].) For other articles on the problem of gravitational radiation from general and binary point-mass sources, see [35–39].

B. Field equations and the no-incoming radiation condition

The problem is to find the solutions, in the form of analytic approximations, of the Einstein field equations in $\mathbb{R}^4$,

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = \frac{8\pi G}{c^4}T^{\mu\nu} \ , \tag{1.1}$$

and thus also of their consequence, the equations of motion of the matter source, $\nabla_\nu T^{\mu\nu} = 0$. Throughout this work we assume the existence and unicity of a global harmonic (or de Donder) coordinate system. This means that we can choose the gauge condition

$$\partial_\nu h^{\mu\nu} = 0 \ ; \qquad h^{\mu\nu} \equiv \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu} \ , \tag{1.2}$$

where g and $g^{\mu\nu}$ denote the determinant and inverse of the covariant metric $g_{\mu\nu}$, and where $\eta^{\mu\nu}$ is an auxiliary flat metric [i.e. $\eta^{\mu\nu} = \mathrm{diag}(-1,1,1,1) = \eta_{\mu\nu}$]. The Einstein field equations (1) can then be replaced by the *relaxed* equations

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4}\tau^{\mu\nu} \ , \tag{1.3}$$

where the box operator is the flat d'Alembertian, $\Box \equiv \Box_\eta = \eta^{\mu\nu}\partial_\mu\partial_\nu$, and where the source term is the sum of a matter part and a gravitational part,

$$\tau^{\mu\nu} \equiv |g|T^{\mu\nu} + \frac{c^4}{16\pi G}\Lambda^{\mu\nu} \ . \tag{1.4}$$

In harmonic coordinates the field equations take the form of simple wave equations, but whose source term is actually a complicated functional of the gravitational field $h^{\mu\nu}$; notably the gravitational part depends on $h^{\mu\nu}$ and its first and second space-time derivatives:

$$\begin{aligned}\Lambda^{\mu\nu} = & - h^{\rho\sigma}\partial^2_{\rho\sigma}h^{\mu\nu} + \partial_\rho h^{\mu\sigma}\partial_\sigma h^{\nu\rho} + \frac{1}{2}g^{\mu\nu}g_{\rho\sigma}\partial_\lambda h^{\rho\tau}\partial_\tau h^{\sigma\lambda} \\ & - g^{\mu\rho}g_{\sigma\tau}\partial_\lambda h^{\nu\tau}\partial_\rho h^{\sigma\lambda} - g^{\nu\rho}g_{\sigma\tau}\partial_\lambda h^{\mu\tau}\partial_\rho h^{\sigma\lambda} + g_{\rho\sigma}g^{\lambda\tau}\partial_\lambda h^{\mu\rho}\partial_\tau h^{\nu\sigma} \\ & + \frac{1}{8}(2g^{\mu\rho}g^{\nu\sigma} - g^{\mu\nu}g^{\rho\sigma})(2g_{\lambda\tau}g_{\epsilon\pi} - g_{\tau\epsilon}g_{\lambda\pi})\partial_\rho h^{\lambda\pi}\partial_\sigma h^{\tau\epsilon} \ .\end{aligned} \tag{1.5}$$

The point is that $\Lambda^{\mu\nu}$ is at least quadratic in h, so the relaxed field equations (3) are very naturally amenable to a perturbative non-linear expansion. As an immediate consequence of the gauge condition (2), the right side of the relaxed equations is conserved in the usual sense, and this is equivalent to the equations of motion of matter:

$$\partial_\nu \tau^{\mu\nu} = 0 \quad \Leftrightarrow \quad \nabla_\nu T^{\mu\nu} = 0 \ . \tag{1.6}$$

We refer to $\tau^{\mu\nu}$ as the total stress-energy pseudo-tensor of the matter and gravitational fields in harmonic coordinates. Since the harmonic coordinate condition is Lorentz covariant, $\tau^{\mu\nu}$ is a tensor with respect to Lorentz transformations (but of course not with respect to general diffeomorphisms).

In order to select the physically sensible solution of the field equations in the case of a bounded system, one must choose some boundary conditions at infinity, i.e. the famous no-incoming radiation condition, which ensures that the system is truly isolated (no radiating sources located at infinity). In principle the no-incoming radiation condition is to be formulated at past null infinity $\mathcal{J}^-$. Here, we shall simplify the formulation by taking advantage of the presence of the Minkowski background $\eta_{\mu\nu}$ to define the no-incoming radiation condition with respect to the Minkowskian past null infinity $\mathcal{J}^-_{\rm M}$. Of course, this does not make sense in the exact theory where only exists the metric $g_{\mu\nu}$ and where the metric $\eta_{\mu\nu}$ is fictituous, but within approximate (post-Minkowskian) methods it is legitimate to view the gravitational field as propagating on the flat background $\eta_{\mu\nu}$, since $\eta_{\mu\nu}$ does exist at any finite order of approximation.

We formulate the no-incoming radiation condition in such a way that it suppresses any homogeneous, regular in $\mathbb{R}^4$, solution of the d'Alembertian equation $\Box h = 0$. We have at our disposal the Kirchhoff formula which expresses $h(\mathbf{x}', t')$ in terms of values of $h(\mathbf{x}, t)$ and its derivatives on a sphere centered on $\mathbf{x}'$ with radius $\rho \equiv |\mathbf{x}' - \mathbf{x}|$ and at retarded time $t \equiv t' - \rho/c$:

$$h(\mathbf{x}', t') = \iint \frac{d\Omega}{4\pi}\Big[\frac{\partial}{\partial\rho}(\rho h) + \frac{1}{c}\frac{\partial}{\partial t}(\rho h)\Big](\mathbf{x}, t) \tag{1.7}$$

where $d\Omega$ is the solid angle spanned by the unit direction $(\mathbf{x} - \mathbf{x}')/\rho$. From the Kirchhoff formula we obtain the no-incoming radiation condition as a limit at $\mathcal{J}^-_{\rm M}$, that is $r \to +\infty$ with $t + r/c$ =const (where $r = |\mathbf{x}|$). In fact we obtain two conditions: the main one,

$$\lim_{\substack{r\to+\infty \\ t+r/c={\rm const}}} \Big[\frac{\partial}{\partial r}(r h^{\mu\nu}) + \frac{1}{c}\frac{\partial}{\partial t}(r h^{\mu\nu})\Big](\mathbf{x}, t) = 0 \ , \tag{1.8}$$

and an auxiliary condition, that $r\partial_\lambda h^{\mu\nu}$ should be bounded at $\mathcal{J}^-_{\rm M}$, coming from the fact that ρ in the Kirchhoff formula (7) differs from r [we have $\rho = r - \mathbf{x}'.\mathbf{n} + O(1/r)$ where $\mathbf{n} = \mathbf{x}/r$].

In fact, we adopt in this article a much more restrictive condition of no-incoming radiation, namely that the field is stationary before some finite instant $-\mathcal{T}$ in the past:

$$t \leq -\mathcal{T} \;\Rightarrow\; \frac{\partial}{\partial t}[h^{\mu\nu}(\mathbf{x},t)] = 0 \;. \tag{1.9}$$

In addition we assume that before $-\mathcal{T}$ the field $h^{\mu\nu}(\mathbf{x})$ is of order $O(1/r)$ when $r \to +\infty$. These restrictive conditions are imposed for technical reasons following [23], since they allow constructing rigorously (and proving theorems about) the metric outside some time-like world tube $r \equiv |\mathbf{x}| > \mathcal{R}$. We shall assume that the region $r > \mathcal{R}$ represents the exterior of an actual compact-support system with constant radius $d < \mathcal{R}$ [i.e. d is the maximal radius of the adherence of the compact support of $T^{\mu\nu}(\mathbf{x},t)$, for any time t].

Now if $h^{\mu\nu}$ satisfies for instance (9), so does the pseudo-tensor $\tau^{\mu\nu}$ built on it, and then it is clear that the retarded integral of $\tau^{\mu\nu}$ satisfies itself the same condition. Therefore one infers that the unique solution of the Einstein equation (3) satisfying the condition (9) is

$$h^{\mu\nu} = \frac{16\pi G}{c^4}\Box^{-1}_R \tau^{\mu\nu} \;, \tag{1.10}$$

where the retarded integral takes the standard form

$$(\Box^{-1}_R\tau)(\mathbf{x},t) \equiv -\frac{1}{4\pi}\int\frac{d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\tau\left(\mathbf{x}', t - |\mathbf{x}-\mathbf{x}'|/c\right) \;. \tag{1.11}$$

Notice that since $\tau^{\mu\nu}$ depends on h and its derivatives, the equation (10) is to be viewed rather as an integro-differential equation equivalent to the Einstein equation (3) with no-incoming radiation.

C. Method and general physical picture

We want to describe an isolated system, for instance a "two-body system", in Einstein's theory. We expect (though this is not proved) that initial data sets $g_{\mu\nu}$, $\partial_t g_{\mu\nu}$, ρ, $\mathbf{v}$ satisfying the constraint equations on the space-like hypersurface $t = t_0$ exist, and that this determines a unique solution of the field equations for any time t, which approaches in the case of two bodies a "scattering state" when $t \to -\infty$, in which the bodies move on unbound (hyperbolic-like) orbits. We assume that the space-times generated by such data admit a past null infinity $\mathcal{J}^-$ (or, if one uses approximate methods, $\mathcal{J}^-_{\rm M}$) with no incoming radiation. (Note that in a situation with initial scattering the field might not satisfy the rigorous definitions of asymptotic flatness at $\mathcal{J}^-$; see [40–43].) The point to make is that in this class of space-times there is no degree of freedom for the gravitational field (we could consider other situations where the motion is influenced by incoming radiation).

Both our technical assumptions of compact support for the matter source (with constant radius d) and stationarity before the time $-\mathcal{T}$ contradict our expectation that a two-body system follows an unbound orbit in the remote past. We do not solve this conflict but argue as follows: (i) these technical assumptions permit to derive rigorously some results, for instance the expression [given by (52) with (56) below] of the far-field of an isolated past-stationary system; (ii) it is clear that these results do not depend on the constant radius d, and furthermore we check that they admit in the "scattering" situation a well-defined limit when $-\mathcal{T} \to -\infty$; (iii) this makes us confident that the results are actually valid for a more realistic class of physical systems which become unbound in the past and are never stationary (and, even, one can give *a posteriori* conditions under which the limit $-\mathcal{T} \to -\infty$ exists for a general system at some order of approximation).

Suppose that the system is "slowly-moving" [in the sense of (12) below], so that we can compute the field inside its compact support by means of a post-Newtonian method, say $h_{\text{in}}^{\mu\nu} \equiv \overline{h}^{\mu\nu}$ where the overbar refers to the *formal* post-Newtonian series. The post-Newtonian iteration (say, for hydrodynamics) is not yet defined to all orders in $1/c$, but many terms are known: see the works of Lorentz and Droste [20], Einstein, Infeld and Hoffmann [21], Fock [44], Chandrasekhar and collaborators [45–47], Ehlers and followers [48–54], and many other authors [55–58,30,24].

On the other hand, outside the isolated system, the field is weak everywhere and it satisfies the vacuum equations. Therefore, the equations can be solved conjointly by means of a weak-field or post-Minkowskian expansion ($G \to 0$), and, for each coefficient of G^n in the latter expansion, by means of a multipole expansion (valid because we are outside). The general Multipolar-post-Minkowskian (MPM) metric was constructed in [23,29] as a functional of two sets of "multipole moments" $M_L(t)$ and $S_L(t)$ which were left arbitrary at this stage (i.e. not connected to the source). The idea of combining the post-Minkowskian and multipole expansions comes from the works of Bonnor [59] and Thorne [60]. We denote by $h_{\text{ext}}^{\mu\nu} \equiv \mathcal{M}(h^{\mu\nu})$ the exterior solution, where $\mathcal{M}$ stands for the multipole expansion (as it will turn out, the post-Minkowskian expansion appears in this formalism to be somewhat less fundamental than the multipole expansion).

The key assumption is that the two expansions $h_{\text{in}}^{\mu\nu} = \overline{h}^{\mu\nu}$ and $h_{\text{ext}}^{\mu\nu} = \mathcal{M}(h^{\mu\nu})$ should match in a region of common validity for both the post-Newtonian and multipole expansions. Here is where our physical restriction to slow motion plays a crucial role, because such an overlap region exists (this is the so-called exterior near-zone) if and only if the system is slowly-moving. The matching is a variant of the well-known method of matching of asymptotic expansions, very useful in gravitational radiation theory [61–65,30,66,24]. It consists of decomposing the inner solution into multipole moments (valid in the outside), re-expanding the exterior solution in the near zone ($r/c \to 0$), and equating term by term the two resulting expansion series. From the requirement of matching we obtain in [25], and review in Sections 2 and 3 below, the general formula for the multipole expansion $\mathcal{M}(h^{\mu\nu})$ in terms of the "source" multipole moments (notably a mass-type moment I_L and a current-type J_L), given as functionals of the *post-Newtonian* expansion of the pseudo-tensor, i.e.

$\overline{\tau}^{\mu\nu}$. [The previous moments M_L and S_L (referred below to as "canonical") are deduced from the source moments after a suitable coordinate transformation.] In addition the matching equation determines the radiation reaction contributions in the inner post-Newtonian metric [67,68,32].

To obtain the source multipole moments in terms of basic source parameters (mass density, pressure), it remains to replace $\overline{\tau}^{\mu\nu}$ by the result of an explicit post-Newtonian iteration of the inner field. This was done to 1PN order in [30,66], then to 2PN order in [24], and the general formulas obtained in [25] permit recovering these results. See Section 6. On the other hand, if one needs the equations of motion of the source, simply one inserts the post-Newtonian metric into the conservation law $\partial_\nu \overline{\tau}^{\mu\nu} = 0$. (Note that we are speaking of the equations of motion, which take for instance the form of Euler-type equations with many relativistic corrections, but not of the *solutions* of these equations, which are typically impossible to obtain analytically.)

From the harmonic coordinates, one can perform to all post-Minkowskian orders [29] a coordinate transformation to some radiative coordinates such that the metric admits a far-field expansion in powers of the inverse of the distance R (without the powers of $\ln R$ which plague the harmonic coordinates). Considering the leading order $1/R$ one compares the exterior metric, which is parametrized by the source moments (connected to the source via the matching equation), to the metric defined with "radiative" multipole moments, say U_L and V_L. This gives U_L and V_L in terms of the source moments, notably I_L and J_L, and *a fortiori* of the source parameters. This solves, within approximate methods, the problem of the relation between the far field and the source. The radiative moments have been obtained with increasing precision reaching now 3PN [69–71], as reviewed in Section 5.

The previous scheme is developed for a general description of matter, however restricted to be smooth (we have in mind a general "hydrodynamical" $T^{\mu\nu}$). Thus the scheme *a priori* excludes the presence of singularities (no "point-particles" or black holes), but this is a serious limitation regarding the application to compact objects like neutron stars, which can adequately be approximated by point-masses when studying their dynamics. Fortunately, the formalism *is* applicable to a singular $T^{\mu\nu}$ involving Dirac measures, at the price of a further ansatz, that the infinite self-field of point-masses can be regularized in a certain way. By implementing consistently the regularization we obtain the multipole moments and the radiation field of a system of two point-masses at 2.5PN order [72,73], as well as their equations of motion at the same order in the form of ordinary differential equations [74] (the result agrees with previous works [16–19]); see Section 7.

II. MULTIPOLE DECOMPOSITION

In this section we construct the multipole expansion $\mathcal{M}(h^{\mu\nu}) \equiv h^{\mu\nu}_{\rm ext}$ of the gravitational field outside an isolated system, supposed to be at once self-gravitating and slowly-moving. By slowly-moving we mean that the typical current and stress densities are small with respect to the energy density, in the sense that

$$\max\left\{\left|\frac{T^{0i}}{T^{00}}\right|,\left|\frac{T^{ij}}{T^{00}}\right|^{1/2}\right\}=O\left(\frac{1}{c}\right)\ , \tag{2.1}$$

where $1/c$ denotes (slightly abusively) the small post-Newtonian parameter. The point about (12) is that the ratio between the size of the source d and a typical wavelength of the gravitational radiation is of order $d/\lambda = O(1/c)$. Thus the domain of validity of the post-Newtonian expansion covers the source: it is given by $r < b$ where the radius b can be chosen so that $d < b = O(\lambda/c)$.

A. The matching equation

The construction of the multipole expansion is based on several technical assumptions, the crucial one being that of the consistency of the asymptotic matching between the exterior and interior fields of the isolated system. In some cases the assumptions can be proved from the properties of the exterior field $h^{\mu\nu}_{\mathrm{ext}}$ as obtained in [23] by means of a post-Minkowskian algorithm. However, since our assumptions are free of any reference to the post-Minkowskian expansion, we prefer to state them more generally, without invoking the existence of such an approximation (refer to [25] for the full detailed assumptions). In many cases the assumptions have been explicitly verified at some low post-Newtonian orders [30,66,24,73].

The field h (skipping space-time indices), solution in $\mathbb{R}^4$ of the relaxed field equations and the no-incoming radiation condition, is given as the retarded integral (10). We now *assume* that outside the isolated system, say, in the region $r > \mathcal{R}$ where $\mathcal{R}$ is a constant radius strictly larger than d, we have $h = \mathcal{M}(h)$ where $\mathcal{M}(h)$ denotes the multipole expansion of h, a solution of the *vacuum* field equations in $\mathbb{R}^4$ deprived from the spatial origin $r = 0$, and admitting a spherical-harmonics expansion of a certain structure (see below). Thus, in $\mathbb{R} \times \mathbb{R}^3_*$ where $\mathbb{R}^3_* \equiv \mathbb{R}^3 - \{\mathbf{0}\}$,

$$\partial_\nu \mathcal{M}(h^{\mu\nu}) = 0\ , \tag{2.2a}$$

$$\Box \mathcal{M}(h^{\mu\nu}) = \mathcal{M}(\Lambda^{\mu\nu})\ . \tag{2.2b}$$

The source term $\mathcal{M}(\Lambda)$ is obtained from inserting $\mathcal{M}(h)$ in place of h into (5), i.e. $\mathcal{M}(\Lambda) \equiv \Lambda(\mathcal{M}(h))$. [Since the matter tensor has a compact support, $\mathcal{M}(T) = 0$ so that $\mathcal{M}(\tau) = \frac{c^4}{16\pi G}\mathcal{M}(\Lambda)$.] Of course, inside the source (when $r \leq d$), the true solution h differs from the vacuum solution $\mathcal{M}(h)$, the latter becoming in fact singular at the origin ($r = 0$). We assume that the spherical-harmonics expansion of $\mathcal{M}(h)$ in $\mathbb{R} \times \mathbb{R}^3_*$ reads

$$\mathcal{M}(h)(\mathbf{x},t) = \sum_{a\leq N} \hat{n}_L r^a (\ln r)^p {}_L F_{a,p}(t) + R_N(\mathbf{x},t)\ . \tag{2.3}$$

This expression is valid for any $N \in \mathbb{N}$. The powers of r are positive or negative, $a \in \mathbb{Z}$, and we have $a \leq N$ (the negative powers of r show that the multipole expansion is singular at $r = 0$). For ease of notation we indicate only the summation over a, but there are two other summations involved: one over the powers $p \in \mathbb{N}$ of the logarithms, and one over the order of multipolarity $l \in I\!N$. The summations are considered only in the sense of formal series, as we do not control the mathematical nature of the series. The factor $\hat{n}_L$ is a product of l unit vectors, $n_L \equiv n^L \equiv n^{i_1}...n^{i_l}$, where $L \equiv i_1...i_l$ is a multi-index with l indices, on which the symmetric and trace-free (STF) projection is applied: $\hat{n}_L \equiv \mathrm{STF}[n_L]$. The decomposition in terms of STF tensors $\hat{n}_L(\theta, \varphi)$ is equivalent to the decomposition in usual spherical harmonics. The functions ${}_LF_{a,p}(t)$ are smooth (C^∞) functions of time, which become constant when $t \leq -\mathcal{T}$ because of our assumption (9). [Of course, the ${}_LF_{a,p}$'s depend also on c: ${}_LF_{a,p}(t, c)$.] Finally the function $R_N(\mathbf{x}, t)$ is defined by continuity throughout $I\!R^4$. Its two essential properties are $R_N \in C^N(I\!R^4)$ and $R_N = O(r^N)$ when $r \to 0$ with fixed t. In addition R_N is zero before the time $-\mathcal{T}$. Though the function $R_N(\mathbf{x}, t)$ is given "globally" (as is the multipole expansion), it represents a small remainder $O(r^N)$ in the expansion of $\mathcal{M}(h)$ when $r \to 0$, which is to be identified with the "near-zone" expansion of the field outside the source. It is convenient to introduce a special notation for the formal near-zone expansion (valid to any order N):

$$\overline{\mathcal{M}(h)}(\mathbf{x}, t) = \sum \hat{n}_L r^a (\ln r)^p {}_LF_{a,p}(t) \ , \tag{2.4}$$

where the summation is to be understood in the sense of formal series. [Note that (14) and (15) are written for the field variable $\mathcal{M}(h)$, but it is easy to check that the same type of structure holds also for the source term $\mathcal{M}(\Lambda)$.]

Our justification of the assumed structure (14) is that it has been *proved* to hold for metrics in the class of Multipolar-post-Minkowskian (MPM) metrics considered in [23], i.e. formal series $h_{\mathrm{ext}} = \sum G^n h_n$ which satisfy the vacuum equations, are stationary in the past, and depend on a *finite* set of independent multipole moments. More precisely, from the theorem 4.1 in [23], the general MPM metric h_{ext}, that we identify in this paper with $\mathcal{M}(h^{\mu\nu})$, is such that the property (14) holds for the h_n's to any order n, with the only difference that to any finite order n the integers a, p, l vary into some finite ranges, namely $a_{\mathrm{min}}(n) \leq a \leq N$, $0 \leq p \leq n-1$ and $0 \leq l \leq l_{\mathrm{max}}(n)$, with $a_{\mathrm{min}}(n) \to -\infty$ and $l_{\mathrm{max}}(n) \to +\infty$ when $n \to +\infty$. The functions ${}_LF_{a,p}$ and the remainder R_N in (14) should therefore be viewed as post-Minkowskian series $\sum G^n {}_LF_{a,p,n}$ and $\sum G^n R_{N,n}$. What we have done in writing (14) and (15) is to assume that one can legitimately consider such formal post-Minkowskian series. Note that because the general MPM metric represents the most general solution of the field equations outside the source (Theorem 4.2 in [23]), it is quite appropriate to identify the general multipole expansion $\mathcal{M}(h)$ with the MPM metric h_{ext}. Actually we shall justify this assumption in Section 5 by recovering from $\mathcal{M}(h)$, step by step in the post-Minkowskian expansion, the MPM metric h_{ext}. Because the properties are proved in [23] for any n, and because we consider the formal post-Minkowskian sum, we see that (14)-(15), viewed as if it were "exact", constitutes a quite natural assumption. In particular we have assumed in (14)-(15) that the multipolar series involves an infinite number of independent multipoles. In summary, we give to the properties (14)-(15) a scope larger than the one of MPM expansions (maybe they could be proved for exact solutions), at the price of counting them among our basic assumptions.

The multipole expansion $\mathcal{M}(h)$ is a mathematical solution of the vacuum equations in $I\!R \times I\!R^3_*$, but whose "multipole moments" (the functions ${}_LF_{a,p}$) are not determined in terms of the source parameters. When the isolated system is slowly moving in the sense of (12), there exists an overlapping region between the domains of validity of the post-Newtonian expansion: the "near-zone" $r < b$, where $d < b = O(\lambda/c)$, and of the multipole expansion: the exterior zone $r > \mathcal{R}$. For this to be true it suffices to choose $\mathcal{R}$, which is restricted only to be strictly larger than d, such that $d < \mathcal{R} < b$. We assume that the field h given by (10) admits in the near-zone a formal post-Newtonian expansion, $h = \overline{h}$ when $r < b$. On the other hand, recall that $h = \mathcal{M}(h)$ when $r > \mathcal{R}$. Matching the two asymptotic expansions $\overline{h}$ and $\mathcal{M}(h)$ in the "matching" region $\mathcal{R} < r < b$ means that the (formal) double series obtained by considering the multipole expansion of (all the coefficients of) the post-Newtonian expansion $\overline{h}$ is *identical* to the double series obtained by taking the near-zone expansion of the multipole expansion. [We use the same overbar notation for the post-Newtonian and near-zone expansions because the near-zone expansion ($r/c \to 0$) of the exterior multipolar field is mathematically equivalent to the expansion when $c \to \infty$ with fixed multipole moments.] The resulting matching equation reads

$$\overline{\mathcal{M}(h)} = \mathcal{M}(\overline{h}) \ . \tag{2.5}$$

This equation should be true term by term, after both sides of the equation are re-arranged as series corresponding to the same expansion parameter. Though looking quite reasonable (if the theory makes sense), the matching equation cannot be justified presently with full generality; however up to 2PN order it was shown to determine a unique solution valid everywhere inside and outside the source [30,66,24]. The matching assumption complements the framework of MPM approximations [23], by giving physical "pith" to the arbitrary multipole moments used in the construction of MPM metrics (see Section 4).

B. The field in terms of multipole moments

Let us consider the relaxed vacuum Einstein equation (13b), whose source term $\mathcal{M}(\Lambda)$, according to our assumptions, owns the structure (14) [recall that (14) applies to $\mathcal{M}(h)$ as well as $\mathcal{M}(\Lambda)$]. We obtain a *particular* solution of this equation (in $I\!R \times I\!R^3_*$) as follows. First we multiply each term composing $\mathcal{M}(\Lambda)$ in (14) by a factor $(r/r_0)^B$, where B is a complex number and r_0 a constant with the dimension of a length. For each term we can choose the real part of B large enough so that the term becomes regular when $r \to 0$, and then we can apply the retarded integral (11). The resulting B-dependent retarded integral is known to be analytically continuable for any $B \in \mathbb{C}$ except at integer values including in general the value of interest $B = 0$. Furthermore one can show that the finite part (in short $\mathrm{FP}_{B=0}$) of this integral, defined to be the coefficient of the zeroth power of B in the expansion when $B \to 0$, is a retarded solution of the corresponding wave equation. In the case of a regular term in (14) such as the remainder R_N, this solution simply reduces to the retarded integral. Summing all these solutions, corresponding to all the separate terms in (14), we thereby obtain as a particular solution of (13b) the object $\mathrm{FP}_{B=0}\,\Box_R^{-1}[(r/r_0)^B\mathcal{M}(\Lambda)]$. This is basically the method employed in [23] to solve the vacuum field equations in the post-Minkowskian approximation.

Now all the problem is to find *the* homogeneous solution to be added to the latter particular solution in order that the multipole expansion $\mathcal{M}(h)$ matches with the post-Newtonian expansion $\overline{h}$, solution within the source of the field equation (3) [or, rather, (10)]. Finding this homogeneous solution means finding the general consequence of the matching equation (16). The result [24,25] is that the multipole expansion $h^{\mu\nu}$ satisfying the Einstein equation (10) together with the matching equation (16) reads

$$\mathcal{M}(h^{\mu\nu}) = \mathrm{FP}_{B=0}\,\Box_R^{-1}[(r/r_0)^B\mathcal{M}(\Lambda^{\mu\nu})] - \frac{4G}{c^4}\sum_{l=0}^{+\infty}\frac{(-)^l}{l!}\partial_L\left\{\frac{1}{r}\mathcal{H}_L^{\mu\nu}(t-r/c)\right\} \tag{2.6}$$

where the first term is the previous particular solution, and where the second term is a retarded solution of the source-free (homogeneous) wave equation, whose "multipole moments" are given explicitly by ($u\equiv t-r/c$)

$$\mathcal{H}_L^{\mu\nu}(u) = \mathrm{FP}_{B=0}\int d^3\mathbf{x}\;|\mathbf{x}/r_0|^B x_L\,\overline{\tau}^{\mu\nu}(\mathbf{x},u)\;. \tag{2.7}$$

Here $\overline{\tau}^{\mu\nu}$ denotes the post-Newtonian expansion of the stress-energy pseudo-tensor $\tau^{\mu\nu}$ appearing in the right side of (10). In (17) and (18) we denote $L=i_1\dots i_l$ and $\partial_L\equiv\partial_{i_1}\dots\partial_{i_l}$, $x_L\equiv x_{i_1}\dots x_{i_l}$.

It is important that the multipole moments (18) are found to depend on the *post-Newtonian* expansion $\overline{\tau}^{\mu\nu}$ of the pseudo-tensor, and not of $\tau^{\mu\nu}$ itself, as this is precisely where our assumption of matching to the inner post-Newtonian field comes in. The formula is *a priori* valid only in the case of a slowly-moving source; it is *a priori* true only after insertion of a definite post-Newtonian expansion of the pseudo-tensor, where in particular all the retardations have been expanded when $c\to\infty$ [the formulas (17)-(18) assume implicitly that one can effectively construct such a post-Newtonian expansion].

Like in the first term of (17), the moments (18) are endowed with a finite part operation defined by complex analytic continuation in B. Notice however that the two finite part operations in the first term of (17) and in (18) act quite differently. In the first term of (17) the analytic continuation serves at regularizing the singularity of the multipole expansion at the spatial *origin* $r=0$. Since the pseudo-tensor is smooth inside the source, there is no need in the moments (18) to regularize the field near the origin; still the finite part is essential because it applies to the bound of the integral at *infinity* ($|\mathbf{x}|\to\infty$). Otherwise the integral would be (*a priori*) divergent at infinity, because of the presence of the factor $x_L=O(r^l)$ in the integrand, and the fact that the pseudo-tensor $\overline{\tau}^{\mu\nu}$ is non-compact supported. The two finite parts present in the two separate terms of (17) involve the same arbitrary constant r_0, but this constant can be readily checked to cancel out between the two terms [i.e. the differentiation of $\mathcal{M}(h^{\mu\nu})$ with respect to r_0 yields zero].

The formulas (17)-(18) were first obtained (in STF form) up to the 2PN order in [24] by performing explicitly the matching. This showed in particular that the matching equation

(16) is correct to 2PN order. Then the proof valid to any post-Newtonian order, but at the price of *assuming* (16) to all orders, was given in Section 3 of [25] (see also Appendix A of [25] for an alternative proof). The crucial step in the proof is to remark that the finite part of the integral of $\overline{\mathcal{M}(\Lambda)}$ over the *whole* space $I\!\!R^3$ is identically zero by analytic continuation:

$$\mathrm{FP}_{B=0} \int_{I\!\!R^3} d^3\mathbf{x}\ |\mathbf{x}/r_0|^B x_L \overline{\mathcal{M}(\Lambda)}(\mathbf{x}, u) = 0\ . \tag{2.8}$$

This follows from the fact that $\overline{\mathcal{M}(\Lambda)}$ can be written as a formal series of the type (15). Using (15) it is easy to reduce the computation of the integral (19) to that of the elementary radial integral $\int_0^{+\infty} d|\mathbf{x}||\mathbf{x}|^{B+2+l+a}$ (since the powers of the logarithm can be obtained by repeatedly differentiating with respect to B). The latter radial integral can be split into a "near-zone" integral, extending from zero to radius $\mathcal{R}$, and a "far-zone" integral, extending from $\mathcal{R}$ to infinity (actually any finite non-zero radius fits instead of $\mathcal{R}$). When the real part of B is a large enough positive number, the value of the near-zone integral is $\mathcal{R}^{B+3+l+a}/(B+3+l+a)$, while when the real part of B is a large *negative* number, the far-zone integral reads the opposite, $-\mathcal{R}^{B+3+l+a}/(B+3+l+a)$. Both obtained values represent the unique analytic continuations of the near-zone and far-zone integrals for any $B \in \mathbb{C}$ except $-3-l-a$. The complete integral $\int_0^{+\infty} d|\mathbf{x}||\mathbf{x}|^{B+2+l+a}$ is defined as the sum of the analytic continuations of the near-zone and far-zone integrals, and is therefore identically zero ($\forall B \in \mathbb{C}$); this proves (19).

One may ask why the whole integration over $I\!\!R^3$ contributes to the multipole moment (18) – a somewhat paradoxical fact because the integrand is in the form of a post-Newtonian expansion, and is thus expected to be physically valid (i.e. to give accurate results) only in the near zone. This fact is possible thanks to the technical identity (19) which enables us to transform a near-zone integration into a complete $I\!\!R^3$-integration (refer to [25] for details).

C. Equivalence with the Will-Wiseman multipole expansion

Recently a different expression of the multipole decomposition, with correlatively a different expression of the multipole moments, was obtained by Will and Wiseman [75], extending previous work of Epstein and Wagoner [76] and Thorne [60]. Basically, the multipole moments in [75] are defined by an integral extending over a ball of *finite* radius $\mathcal{R}$ (essentially the same $\mathcal{R}$ as here), and thus do not require any regularization of the bound at infinity. By contrast, our multipole moments (18) involve an integration over the whole $I\!\!R^3$, which is allowed thanks to the analytic continuation [leading to the identity (19)]. Let us outline the proof of the equivalence between the Will-Wiseman formalism [75] and the present one [24,25].

Will and Wiseman [75] find, instead of (17)-(18),

$$\mathcal{M}(h^{\mu\nu}) = \Box^{-1}_R[\mathcal{M}(\Lambda^{\mu\nu})]_{|\mathcal{R}} - \frac{4G}{c^4}\sum_{l=0}^{+\infty}\frac{(-)^l}{l!}\partial_L\left\{\frac{1}{r}\mathcal{W}^{\mu\nu}_L(t-r/c)\right\}\ . \tag{2.9}$$

The first term is given by the retarded integral (11) acting on $\mathcal{M}(\Lambda)$, but *truntated*, as indicated by the subscript $\mathcal{R}$, to extend only in the "far zone": $|\mathbf{x}'| > \mathcal{R}$ in the notation (11). Thus, the near-zone part of the retarded integral, which contains the source, is removed, and there is no problem with the singularity of the multipole expansion at the origin. Then, the multipole moments $\mathcal{W}_L$ are given by an integral extending over the "near zone" only:

$$\mathcal{W}_L^{\mu\nu}(u) = \int_{|\mathbf{x}|<\mathcal{R}} d^3\mathbf{x}\; x_L\,\overline{\tau}^{\mu\nu}(\mathbf{x},u)\ . \tag{2.10}$$

The integral being compact-supported is well-defined. The multipole moments $\mathcal{W}_L$ look technically more simple than ours given by (18). On the other hand, practically speaking, the analytic continuation in (18) permits deriving many closed-form formulas to be used in applications [72,77]. Of course, one is free to choose any definition of the multipole moments as far as it is used in a consistent manner.

We compute the difference between the moments $\mathcal{H}_L$ and $\mathcal{W}_L$. For the comparison we split $\mathcal{H}_L$ into far-zone and near-zone integrals corresponding to the radius $\mathcal{R}$. Since the analytic continuation factor in $\mathcal{H}_L$ deals only with the bound at infinity, it can be removed from the near-zone integral, which is then clearly seen to agree with $\mathcal{W}_L$. So the difference $\mathcal{H}_L - \mathcal{W}_L$ is given by the far-zone integral:

$$\mathcal{H}_L(u) - \mathcal{W}_L(u) = \mathrm{FP}_{B=0}\int_{|\mathbf{x}|>\mathcal{R}} d^3\mathbf{x}\; |\mathbf{x}/r_0|^B x_L\overline{\tau}(\mathbf{x},u)\ . \tag{2.11}$$

Next we transform the integrand. Successively we write $\overline{\tau} = \mathcal{M}(\overline{\tau})$ because we are in the far zone; $\mathcal{M}(\overline{\tau}) = \overline{\mathcal{M}(\tau)}$ from the matching equation (16); and $\overline{\mathcal{M}(\tau)} = \frac{c^4}{16\pi G}\overline{\mathcal{M}(\Lambda)}$ because T has a compact support. At this stage, the technical identity (19) allows one to transform the far-zone integration into a near zone integration (changing simply the overall sign in front of the integral). So,

$$\mathcal{H}_L(u) - \mathcal{W}_L(u) = -\frac{c^4}{16\pi G}\mathrm{FP}_{B=0}\int_{|\mathbf{x}|<\mathcal{R}} d^3\mathbf{x}\; |\mathbf{x}/r_0|^B x_L\overline{\mathcal{M}(\Lambda)}(\mathbf{x},u)\ . \tag{2.12}$$

It is straightforward to check that the right side of this equation, when summed up over all multipolarities l, accounts exactly for the near-zone part that was removed from the retarded integral of $\mathcal{M}(\Lambda)$ [first term in (20)], so that the "complete" retarded integral as given by the first term in (17) is exactly reconstituted. In conclusion the two formalisms [24,25] and [75] are equivalent.

III. SOURCE MULTIPOLE MOMENTS

Quite naturally our source multipole moments will be closely related to the $\mathcal{H}_L$'s obtained in (18). However, before giving a precise definition, we need to find the equivalent of the multipole decomposition (17)-(18) in terms of symmetric and trace-free (STF) tensors, and we must reduce the number of independent tensors by imposing the harmonic gauge condition (13a). This leads to the definition of a "linearized" metric associated with the multipole expansion $\mathcal{M}(h)$, and parametrized by six sets of STF source multipole moments.

A. Multipole expansion in symmetric-trace-free form

The moments $\mathcal{H}_L$ given by (18) are non-trace-free because x_L owns all its traces (i.e. $\delta_{i_l i_{l-1}} x_L = \mathbf{x}^2 x_{L-2}$, where $L-2 = i_1 ... i_{l-2}$). Instead of $\mathcal{H}_L$, there are certain advantages in using STF multipole moments: indeed the STF moments are uniquely defined, and they often yield simpler computations in practice. It is not difficult, using STF techniques, to obtain the multipole decomposition equivalent to (17)-(18) but expressed in terms of STF tensors. We find

$$\mathcal{M}(h^{\mu\nu}) = \mathrm{FP}_{B=0} \Box_R^{-1}[(r/r_0)^B \mathcal{M}(\Lambda^{\mu\nu})] - \frac{4G}{c^4} \sum_{l=0}^{+\infty} \frac{(-)^l}{l!} \partial_L \left\{ \frac{1}{r} \mathcal{F}_L^{\mu\nu}(t - r/c) \right\} \tag{3.1}$$

where the STF multipole moments are given by [25]

$$\mathcal{F}_L^{\mu\nu}(u) = \mathrm{FP}_{B=0} \int d^3\mathbf{x} \, |\mathbf{x}/r_0|^B \hat{x}_L \int_{-1}^{1} dz \; \delta_l(z) \overline{\tau}^{\mu\nu}(\mathbf{x}, u + z|\mathbf{x}|/c) \; . \tag{3.2}$$

The notation for a STF product of vectors is $\hat{x}_L \equiv \mathrm{STF}(x_L)$ (such that $\hat{x}_L$ is symmetric in L and $\delta_{i_l i_{l-1}} \hat{x}_L = 0$; for instance $\hat{x}_{ij} = x_i x_j - \frac{1}{3}\delta_{ij}\mathbf{x}^2$). As we see, the STF moments (25) involve an extra integration, over the variable z, with respect to the non-STF ones (18). The weighting function associated with the z-integration reads, for any l,

$$\delta_l(z) = \frac{(2l+1)!!}{2^{l+1} l!}(1-z^2)^l \; ; \quad \int_{-1}^{1} dz \; \delta_l(z) = 1 \; . \tag{3.3}$$

In the limit of large l the weighting function tends toward the Dirac delta measure (hence its name): $\lim_{l\to\infty} \delta_l = \delta$. Remark that since (25) is valid only in the post-Newtonian approximation, the z-integration is to be expressed as a post-Newtonian series. Here is the relevant formula [30]:

$$\int_{-1}^{1} dz \; \delta_l(z) \overline{\tau}(\mathbf{x}, u + z|\mathbf{x}|/c) = \sum_{k=0}^{\infty} \frac{(2l+1)!!}{2^k k! (2l+2k+1)!!} \left(\frac{|\mathbf{x}|}{c} \frac{\partial}{\partial u} \right)^{2k} \overline{\tau}(\mathbf{x}, u) \; . \tag{3.4}$$

In the limiting case of linearized gravity, one can neglect the first term in (24), and the pseudo-tensor $\overline{\tau}^{\mu\nu}$ in (25) can be replaced by the matter stress-energy tensor $T^{\mu\nu}$ (we have $\overline{T}^{\mu\nu} = T^{\mu\nu}$ inside the slowly-moving source). Since $T^{\mu\nu}$ has a compact support the finite part prescription can be removed, and we recover the known multipole decomposition corresponding to a compact-support source (see the appendix B in [30]).

B. Linearized approximation to the exterior field

Up to now we have solved the *relaxed* field equation (10) in the exterior zone, with result the multipole decomposition (24)-(25). In this section we further impose the harmonic gauge condition (13a), and from this we find a solution of the linearized vacuum equation, appearing as the first approximation in a post-Minkowskian expansion of the multipole expansion $\mathcal{M}(h)$.

Let us give a notation to the first term in (24):

$$u^{\mu\nu} \equiv \mathrm{FP}_{B=0}\,\Box_R^{-1}[(r/r_0)^B\mathcal{M}(\Lambda^{\mu\nu})]\ . \tag{3.5}$$

Applying on (24) the condition $\partial_\nu\mathcal{M}(h^{\mu\nu})=0$, we find that the divergence $w^\mu\equiv\partial_\nu u^{\mu\nu}$ is equal to a retarded solution of the source-free wave equation, given by

$$w^\mu = \frac{4G}{c^4}\partial_\nu\Big(\sum_{l=0}^{+\infty}\frac{(-)^l}{l!}\partial_L\left\{\frac{1}{r}\mathcal{F}_L^{\mu\nu}(t-r/c)\right\}\Big)\ . \tag{3.6}$$

Now, associated to any w^μ of this type, there exists some $v^{\mu\nu}$ which is like w^μ a retarded solution of the source-free wave equation, $\Box(v^{\mu\nu})=0$, and furthermore whose divergence is the opposite of w^μ, $\partial_\nu v^{\mu\nu}=-w^\mu$. We refer to [23,70] for the explicit formulas allowing the "algorithmic" construction of $v^{\mu\nu}$ once we know w^μ. For definiteness, we adopt the formulas (2.12) in [70], which represent themselves a slight modification of the earlier formulas (4.13) in [23] (see also the appendix B in [25]).

With $v^{\mu\nu}$ at our disposal we define what constitutes the linearized approximation to the exterior metric, say $Gh_1^{\mu\nu}$ where we factorize out G in front of the metric in order to emphasize its linear character:

$$Gh_1^{\mu\nu} \equiv -\frac{4G}{c^4}\sum_{l=0}^{+\infty}\frac{(-)^l}{l!}\partial_L\left\{\frac{1}{r}\mathcal{F}_L^{\mu\nu}(t-r/c)\right\} - v^{\mu\nu}\ . \tag{3.7}$$

The linearized metric satisfies the linearized vacuum equations in harmonic gauge: $\Box h_1^{\mu\nu}=0$ since both terms in (30) satisfy the source-free wave equation, and $\partial_\nu h_1^{\mu\nu}=0$ thanks to (29) and $\partial_\nu v^{\mu\nu}=-w^\mu$. Using the definition (30) one can re-write the multipole expansion of the exterior field as

$$\mathcal{M}(h^{\mu\nu}) = Gh_1^{\mu\nu}+u^{\mu\nu}+v^{\mu\nu}\ . \tag{3.8}$$

Quite naturally the $u^{\mu\nu}$ and $v^{\mu\nu}$ will represent the *non-linear* corrections to be added to the "linearized" metric $Gh_1^{\mu\nu}$ in order to reconstruct the complete exterior metric (see Section 4).

Since $h_1^{\mu\nu}$ satisfies $\Box h_1^{\mu\nu}=0=\partial_\nu h_1^{\mu\nu}$, there is a unique way to decompose it into the sum of a "canonical" metric introduced by Thorne [60] (see also [23]) plus a linearized gauge transformation,

$$h_1^{\mu\nu} = h_{\mathrm{can}1}^{\mu\nu}+\partial^\mu\varphi_1^\nu+\partial^\nu\varphi_1^\mu-\eta^{\mu\nu}\partial_\lambda\varphi_1^\lambda\ . \tag{3.9}$$

The canonical linearized metric is defined by

$$h^{00}_{\rm can1} = -\frac{4}{c^2}\sum_{l\geq 0}\frac{(-)^l}{l!}\partial_L\left(\frac{1}{r}I_L(u)\right) \ , \tag{3.10a}$$

$$h^{0i}_{\rm can1} = \frac{4}{c^3}\sum_{l\geq 1}\frac{(-)^l}{l!}\left\{\partial_{L-1}\left(\frac{1}{r}I^{(1)}_{iL-1}(u)\right) + \frac{l}{l+1}\varepsilon_{iab}\partial_{aL-1}\left(\frac{1}{r}J_{bL-1}(u)\right)\right\} \ , \tag{3.10b}$$

$$h^{ij}_{\rm can1} = -\frac{4}{c^4}\sum_{l\geq 2}\frac{(-)^l}{l!}\left\{\partial_{L-2}\left(\frac{1}{r}I^{(2)}_{ijL-2}(u)\right) + \frac{2l}{l+1}\partial_{aL-2}\left(\frac{1}{r}\varepsilon_{ab(i}J^{(1)}_{j)bL-2}(u)\right)\right\} \ , \tag{3.10c}$$

where the I_L's and J_L's are two sets of functions of the retarded time $u = t - r/c$ [the subscript (n) indicates n time derivatives], and which are STF with respect to all their indices $L = i_1 \dots i_l$ (the symmetrization is denoted with parenthesis). As for the gauge vector φ^μ_1, it satisfies $\Box\varphi^\mu_1 = 0$ and depends in a way similar to (33) on four other sets of STF functions of u, denoted W_L, X_L, Y_L and Z_L (one type of function for each component of the vector). See [25] for the expression of $\varphi^\mu_1 = \varphi^\mu_1[W_L, X_L, Y_L, Z_L]$.

C. Derivation of the source multipole moments

The two sets of multipole moments I_L and J_L parametrizing the metric (33) constitute our definitions for respectively the mass-type and current-type multipole moments of the source. Actually, there are also the moments W_L, X_L, Y_L, Z_L, and we refer collectively to $\{I_L, J_L, W_L, X_L, Y_L, Z_L\}$ as the set of six *source* multipole moments.

With (32) it is easily seen (because $\Box\varphi^\mu_1 = 0$) that the gauge condition $\partial_\nu h^{\mu\nu}_1 = 0$ imposes no condition on the source moments except the conservation laws appropriate to the gravitational monopole I (having $l = 0$) and dipoles I_i, J_i ($l = 1$): namely,

$$I^{(1)} = 0 \ ; \qquad I^{(2)}_i = 0 \ ; \qquad J^{(1)}_i = 0 \ . \tag{3.11}$$

The mass monopole I and current dipole J_i are thus constant, and agree respectively with the ADM mass and total angular momentum of the isolated system (later we shall denote the ADM mass by $M \equiv I$). According to (34) the mass dipole I_i is a linear function of time, but since we assumed that the metric is stationary in the past, I_i is in fact also constant, and equal to the (ADM) center of mass position.

The expressions of I_L and J_L (as well as of the other moments W_L, X_L, Y_L, Z_L) come directly from (30) with (32)-(33) and the result of the matching, which is personified by the formula (25). To simplify the notation we define

$$\Sigma \equiv \frac{\overline{\tau}^{00} + \overline{\tau}^{ii}}{c^2} \ , \tag{3.12a}$$

$$\Sigma_i \equiv \frac{\overline{\tau}^{0i}}{c} \ , \tag{3.12b}$$

$$\Sigma_{ij} \equiv \overline{\tau}^{ij} \ , \tag{3.12c}$$

(where $\overline{\tau}^{ii} \equiv \delta_{ij}\overline{\tau}^{ij}$). The result is [25]

$$I_L(u) = \mathrm{FP}_{B=0} \int d^3\mathbf{x}\ |\mathbf{x}/r_0|^B \int_{-1}^{1} dz \Big\{ \delta_l \hat{x}_L \Sigma - \frac{4(2l+1)}{c^2(l+1)(2l+3)} \delta_{l+1} \hat{x}_{iL} \partial_t \Sigma_i$$
$$+ \frac{2(2l+1)}{c^4(l+1)(l+2)(2l+5)} \delta_{l+2} \hat{x}_{ijL} \partial_t^2 \Sigma_{ij} \Big\} (\mathbf{x}, u + z|\mathbf{x}|/c) \ , \tag{3.13a}$$

$$J_L(u) = \varepsilon_{ab<i_l} \mathrm{FP}_{B=0} \int d^3\mathbf{x}\ |\mathbf{x}/r_0|^B \int_{-1}^{1} dz \Big\{ \delta_l \hat{x}_{L-1>a} \Sigma_b$$
$$- \frac{2l+1}{c^2(l+2)(2l+3)} \delta_{l+1} \hat{x}_{L-1>ac} \partial_t \Sigma_{bc} \Big\} (\mathbf{x}, u + z|\mathbf{x}|/c) \ , \tag{3.13b}$$

($<>$ refers to the STF projection). In a sense these expressions are *exact*, since they are formally valid up to any post-Newtonian order. [See (68)-(69) below for explicit formulas at 2PN.]

By replacing $\overline{\tau}^{\mu\nu}$ in (36) by the compact-support matter tensor $T^{\mu\nu}$ we recover the expressions of the multipole moments worked out in linearized gravity by Damour and Iyer [78] (see also [79]). On the other hand the formulas (36) contain the results obtained by explicit implementation ("order by order") of the matching up to the 2PN order [24].

IV. POST-MINKOWSKIAN APPROXIMATION

In linearized gravity, the source multipole moments represent also the moments which are "measured" at infinity, using an array of detectors surrounding the source. However, in the non-linear theory, the gravitational source $\Lambda^{\mu\nu}$ cannot be neglected and the first term in (24) plays a crucial role, notably it implies that the measured multipole moments at infinity differ from the source moments. Thus, we must now supplement the formulas of the source multipole moments (36) by the study of the "non-linear" term $u^{\mu\nu} \equiv \mathrm{FP}_{B=0}\Box^{-1}_R[(r/r_0)^B \mathcal{M}(\Lambda^{\mu\nu})]$ in (24). For this purpose we develop following [23] a post-Minkowskian approximation for the exterior vacuum metric.

A. Multipolar-post-Minkowskian iteration of the exterior field

The work started already with the formulas (31)-(33), where we expressed the exterior multipolar metric $h^{\mu\nu}_{\rm ext} \equiv \mathcal{M}(h^{\mu\nu})$ as the sum of the "linearized" metric $Gh^{\mu\nu}_1$ and the "non-linear" corrections $u^{\mu\nu}$, given by (28), and $v^{\mu\nu}$, algorithmically constructed from $w^\mu = \partial_\nu u^{\mu\nu}$ [see (29)]. The linearized metric is a functional of the source multipole moments: $h_1 = h_1[I, J, W, X, Y, Z]$. We regard G as the book-keeping parameter for the post-Minkowskian series, and consider that Gh_1 is purely of first order in G, and thus that h_1 itself is purely of zeroth order. Of course we know from the previous section that this is untrue, because the source multipole moments depend on G; supposing $h_1 = O(G^0)$ is simply a convention allowing the systematic implementation of the post-Minkowskian iteration.

Here we check that the non-linear corrections $u^{\mu\nu}$ and $v^{\mu\nu}$ in (31) generate the whole post-Minkowskian algorithm of [23]. The detail demanding attention is how the post-Minkowskian expansions of $u^{\mu\nu}$ and $v^{\mu\nu}$ are related to a spliting of the gravitational source $\Lambda^{\mu\nu}$ into successive non-linear terms. Let us pose, with obvious notation,

$$\Lambda^{\mu\nu} = N^{\mu\nu}[h,h] + M^{\mu\nu}[h,h,h] + O(h^4) \; , \tag{4.1}$$

where, from the exact formula (5), the quadratic-order piece reads (all indices being lowered with the Minkowski metric, and h denoting $\eta^{\rho\sigma}h_{\rho\sigma}$):

$$\begin{aligned} N^{\mu\nu}[h,h] = \; & -h^{\rho\sigma}\partial^2_{\rho\sigma}h^{\mu\nu} + \frac{1}{2}\partial^\mu h_{\rho\sigma}\partial^\nu h^{\rho\sigma} - \frac{1}{4}\partial^\mu h \partial^\nu h \\ & - \partial^\mu h_{\rho\sigma}\partial^\rho h^{\nu\sigma} - \partial^\nu h_{\rho\sigma}\partial^\rho h^{\mu\sigma} + \partial_\sigma h^{\mu\rho}(\partial^\sigma h^\nu_\rho + \partial_\rho h^{\nu\sigma}) \\ & + \eta^{\mu\nu}\left[-\frac{1}{4}\partial_\lambda h_{\rho\sigma}\partial^\lambda h^{\rho\sigma} + \frac{1}{8}\partial_\rho h \partial^\rho h + \frac{1}{2}\partial_\rho h_{\sigma\lambda}\partial^\sigma h^{\rho\lambda}\right] \; , \end{aligned} \tag{4.2}$$

and where the cubic-order piece $M[h,h,h]$ and all higher-order terms can be obtained in a straightforward way.

First, reasoning *ad absurdio*, we prove (see [25] for details) that both u and v indeed represent non-linear corrections to the linearized metric since they start at order G^2: $u = G^2 u_2 + O(G^3)$ and $v = G^2 v_2 + O(G^3)$. Next we obtain explicitly u_2 by substituting the linearized metric h_1 into (38) and applying the finite part of the retarded integral, i.e.

$$u_2^{\mu\nu} = \mathrm{FP}_{B=0}\Box_R^{-1}\Big\{(r/r_0)^B N^{\mu\nu}[h_1,h_1]\Big\} \; . \tag{4.3}$$

In this way we have a particular solution of the wave equation in $I\!R \times I\!R^3_*$, $\Box u_2 = N[h_1,h_1]$. From u_2 one deduces v_2 by the same "algorithmic" equations as used when deducing v from u [see after (29)]. Then $\Box v_2 = 0$ and the sum $u_2 + v_2$ is divergenceless, so we can solve the quadratic-order vacuum equations in harmonic coordinates by posing

$$h_2^{\mu\nu} = u_2^{\mu\nu} + v_2^{\mu\nu} \; . \tag{4.4}$$

With this definition it is clear that the multipole expansion (31) reads to quadratic order:

$$\mathcal{M}(h^{\mu\nu}) = G h_1^{\mu\nu} + G^2 h_2^{\mu\nu} + O(G^3) \; . \tag{4.5}$$

Continuing in this fashion to the next order we find successively

$$u_3^{\mu\nu} = \mathrm{FP}_{B=0}\Box_R^{-1}\Big\{(r/r_0)^B\Big(M^{\mu\nu}[h_1,h_1,h_1] + N^{\mu\nu}[h_1,h_2] + N^{\mu\nu}[h_2,h_1]\Big)\Big\}; \tag{4.6a}$$

$$h_3^{\mu\nu} = u_3^{\mu\nu} + v_3^{\mu\nu}; \tag{4.6b}$$

$$\mathcal{M}(h^{\mu\nu}) = G h_1^{\mu\nu} + G^2 h_2^{\mu\nu} + G^3 h_3^{\mu\nu} + O(G^4) \; . \tag{4.6c}$$

This process continues *ad infinitum*. The latter post-Minkowskian algorithm is exactly the one proposed in [23] (see also Section 2 of [70]). That is, starting from $h_1[I, J, W, X, Y, Z]$ given by (32)-(33), one generates the infinite post-Minkowskian (MPM) series of [23], solving the vacuum (harmonic-coordinate) Einstein equations in $I\!R \times I\!R^3_*$, and this formal series happens to be equal, term by term in G, to the *general* multipole decomposition of $h^{\mu\nu}$ given by (24). For any n, we have $h^{\mu\nu}_n = u^{\mu\nu}_n + v^{\mu\nu}_n$, and

$$\mathcal{M}(h^{\mu\nu}) = \sum_{n=1}^{+\infty} G^n h^{\mu\nu}_n \; . \tag{4.7}$$

This result is perfectly consistent with the fact that the MPM algorithm generates the most *general* solution of the field equations in $I\!R \times I\!R^3_*$. Furthermore, the latter post-Minkowskian approximation is known [26] to be reliable (existence of a one-parameter family of exact solutions whose Taylor expansion when $G \to 0$ reproduces the approximation) – an interesting result which indicates that the multipole decomposition $\mathcal{M}(h)$ given by (24)-(25) might be proved within a context of exact solutions.

Recall that the source multipole moments I_L, J_L, W_L, X_L, Y_L, Z_L entering the linearized metric h_1 at the basis of the post-Minkowskian algorithm are given by formulas like (36). Thus, in the present formalism, the source moments, including formally all post-Newtonian corrections [and all possible powers of G] as contained in (36), serve as "seeds" for the post-Minkowskian iteration of the exterior field, which as it stands leads to all possible non-linear interactions between the moments. As we can imagine, rapidly the formalism becomes extremely complicated when going to higher and higher post-Minkowskian and/or post-Newtonian approximations. Most likely the complexity is not due to the formalism but reflects the complexity of the field equations. It is probably impossible to find a different formalism in which things would be much simpler (except if one restricts to a particular type of source).

B. The "canonical" multipole moments

The previous post-Minkowskian algorithm started with h_1, a functional of *six* types of source multipole moments, I_L and J_L entering the "canonical" linearized metric $h_{\mathrm{can}1}$ given by (33), and W_L, X_L, Y_L, Z_L parametrizing the gauge vector φ_1 in (32). All these moments deserve their name of source moments, but clearly the moments W_L, X_L, Y_L and Z_L do not play a physical role at the level of the linearized approximation, as they simply parametrize a linear gauge transformation. But because the theory is covariant with respect to (non-linear) diffeomorphisms and not merely to linear gauge transformations, these moments do contribute to physical quantities at the non-linear level.

In practice, the presence of the moments W_L, X_L, Y_L, Z_L complicates the post-Minkowskian iteration. Fortunately one can take advantage of the fact (proved in [23]) that it is always possible to parametrize the vacuum metric by means of two and only two types of multipole moments M_L and S_L (different from I_L and J_L). The metric is then obtained by the same post-Minkowskian algorithm as in (39)-(43), but starting with the "canonical" linearized metric $h_{\rm can1}[M,S]$ instead of $h_1[I,J,W,X,Y,Z]$. The resulting non-linear metric $h_{\rm can}$ is isometric to our exterior metric $h_{\rm ext} \equiv \mathcal{M}(h)$, provided that the moments M_L and S_L are given in terms of the source moments $I_L, J_L, \ldots, Z_L$ by some specific relations

$$M_L = M_L[I,J,W,X,Y,Z] \; , \tag{4.8a}$$

$$S_L = S_L[I,J,W,X,Y,Z] \; . \tag{4.8b}$$

The two coordinate systems in which $h_{\rm can}$ and $h_{\rm ext}$ are defined satisfy the harmonic gauge condition in the exterior zone, but (probably) only the one associated with $h_{\rm ext}$ meshes with the harmonic coordinates in the interior zone. With the notation (32) the coordinate change reads $\delta x^\mu = G\varphi_1^\mu +$ non-linear corrections. We shall refer to the moments M_L and S_L as the mass-type and current-type *canonical* multipole moments. Of course, since at the linearized approximation the only "physical" moments are I_L and J_L, we have

$$M_L = I_L + O(G) \; , \tag{4.9a}$$

$$S_L = J_L + O(G) \; , \tag{4.9b}$$

where $O(G)$ denotes the post-Minkowskian corrections. Furthermore, it can be shown [73] that in terms of a post-Newtonian expansion the difference between both types of moments is very small: 2.5PN order, i.e.

$$M_L = I_L + O\left(\frac{1}{c^5}\right) \tag{4.10}$$

[note that $M = M_{\rm ADM} = I$]. Thus, from (46), the canonical moments are only "slightly" different from the source moments. Their usefulness is merely practical – in general they are used in place of the source moments to simplify a computation.

C. Retarded integral of a multipolar extended source

The previous post-Minkowskian algorithm has only theoretical interest unless we supply it with some *explicit* formulas for the computation of the coefficients h_n. Happily for us pragmatists, such formulas exist, and can be found in a rather elegant way thanks to the process of analytic continuation. Basically we need the retarded integral of an extended (non-compact-support) source with a definite multipolarity l. Here we present three exemplifying formulas; see the appendices A in [70] and [71] for more discussion.

Very often we meet a wave equation whose source term is of the type $\hat{n}_L F(t-r/c)/r^k$, where $\hat{n}_L$ has multipolarity l and F denotes a certain product of multipole moments. [Clearly, the near-zone expansion of such a term is of the form (15).] When the power k is such that $3 \le k \le l+2$ (this excludes the scalar case $l=0$), we obtain the solution of the wave equation as [23,68]

$$\mathrm{FP}_{B=0}\Box_R^{-1}\Big[(r/r_0)^B\frac{\hat{n}_L}{r^k}F(t-r/c)\Big] = -\frac{(k-3)!(l+2-k)!}{(l+k-2)!}\hat{n}_L \\ \times\sum_{j=0}^{k-3}2^{k-3-j}\frac{(l+j)!}{j!(l-j)!}\frac{F^{(k-3-j)}(t-r/c)}{c^{k-3-j}\;r^{j+1}}\;. \tag{4.11}$$

As we see the (finite part of the) retarded integral depends in this case on the values of the extended source at the *same* retarded time $t-r/c$ (for simplicity we use the same notation for the source and field points). But it is well known (see e.g. [80,81]) that this feature is exceptional; in most cases the retarded integral depends on the whole integrated past of the source. A chief example of such a "hereditary" character is the case with $k=2$ in the previous example, for which we find [68,69]

$$\Box_R^{-1}\Big[\frac{\hat{n}_L}{r^2}F(t-r/c)\Big] = -\frac{\hat{n}_L}{r}\int_{-\infty}^{ct-r}dsF(s/c)Q_l\Big(\frac{ct-s}{r}\Big) \tag{4.12}$$

where Q_l denotes the Legendre function of the second kind, related to the usual Legendre polynomial P_l by the formula

$$Q_l(x) = \frac{1}{2}P_l(x)\ln\left(\frac{x+1}{x-1}\right) - \sum_{j=1}^{l}\frac{1}{j}P_{l-j}(x)P_{j-1}(x)\;. \tag{4.13}$$

Since the retarded integral (48) is in fact convergent when $r\to 0$, we have removed the factor $(r/r_0)^B$ and finite part prescription. When the source term itself is given by a "hereditary" expression such as the right side of (48), we get a more complicated but still manageable formula, for instance [71]

$$\Box_R^{-1}\Big[\frac{\hat{n}_L}{r^2}\int_{-\infty}^{ct-r}dsF(s/c)Q_p\Big(\frac{ct-s}{r}\Big)\Big] = \frac{c\hat{n}_L}{r}\int_{-\infty}^{ct-r}dsF^{(-1)}(s/c)R_{lp}\Big(\frac{ct-s}{r}\Big) \tag{4.14}$$

where $F^{(-1)}$ denotes that anti-derivative of F which is zero in the past [from (9) we have restricted F to be zero in the past], and where

$$R_{lp}(x) = Q_l(x)\int_1^x dy\,Q_p(y)\frac{dP_l}{dy}(y) + P_l(x)\int_x^{+\infty}dy\,Q_p(y)\frac{dQ_l}{dy}(y)\;. \tag{4.15}$$

Like in (48) we do not need a finite part operation. The function R_{lp} is well-defined thanks to the behaviour of the Legendre function at infinity: $Q_l(x)\sim 1/x^{l+1}$ when $x\to\infty$.

The formulas (48)-(51) are needed to investigate the so-called tails of gravitational waves appearing at quadratic non-linear order, and even the tails generated by the tails themselves ("tails of tails") which arise at cubic order [69,71]. (These formulas do not show a dependence on the constant r_0, but other formulas do.)

V. RADIATIVE MULTIPOLE MOMENTS

In Section 2 we introduced the *definition* of a set of multipole moments $\{I_L, J_L, W_L, X_L, Y_L, Z_L\}$ for the isolated source, and in Section 3 we showed that the exterior field, and in particular the asymptotic field therein, is actually a complicated non-linear functional of the latter moments. Therefore, to define some source multipole moments is not sufficient by itself; this must be completed by a study of the relation between the adopted definition and some convenient far-field observables. The same is true of other definitions of source moments in different formalisms, such as in the Dixon local description of extended bodies [82–84], which should be completed by a connection to the far-zone gravitational field, for instance along the line proposed by [85,86] in the case of the Dixon moments. In the present formalism, the connection rests on the relation between the so-called *radiative* multipole moments, denoted U_L and V_L, and the source moments I_L, J_L,. . ., Z_L [in fact, for simplicity's sake, we prefer using the two moments M_L and S_L instead of the more basic six source moments].

A. Definition and general structure

The radiative moments U_L (mass-type) and V_L (current-type) are the coefficients of the multipolar decomposition of the leading $1/R$ part of the transverse-tracefree (TT) projection of the radiation field in radiative coordinates $(T, \mathbf{X})$ (with $R = |\mathbf{X}|$ the radial distance to the source). Radiative coordinates are such that the metric coefficients admit an expansion when $R \to \infty$ in powers of $1/R$ (no logarithms of R). In radiative coordinates the retarded time $T - R/c$ is light-like, or becomes asymptotically light-like when $R \to \infty$. By *definition*,

$$h_{ij}^{TT}(\mathbf{X}, T) = \frac{4G}{c^2 R}\mathcal{P}_{ijab}(\mathbf{N}) \sum_{l\geq 2} \frac{1}{c^l l!}\Big\{N_{L-2}U_{abL-2} - \frac{2l}{c(l+1)}N_{cL-2}\varepsilon_{cd(a}V_{b)dL-2}\Big\} + O\left(\frac{1}{R^2}\right) , \quad (5.1)$$

where $N_i = X^i/R$, $N_{L-2} = N_{i_1} \dots N_{i_{l-2}}$, $N_{cL-2} = N_c N_{L-2}$, and the TT *algebraic* projector reads $\mathcal{P}_{ijab} = (\delta_{ia} - N_i N_a)(\delta_{jb} - N_j N_b) - \frac{1}{2}(\delta_{ij} - N_i N_j)(\delta_{ab} - N_a N_b)$. The radiative moments U_L and V_L depend on $T - R/c$; from (52) they are defined $\forall\, l \geq 2$. The radiative-coordinate retarded time differs from the corresponding harmonic-coordinate time by the well-known logarithmic deviation of light cones,

$$T - \frac{R}{c} = t - \frac{r}{c} - \frac{2GM}{c^3}\ln\left(\frac{r}{r_0}\right) + O(G^2) , \quad (5.2)$$

where we have introduced in the logarithm the same constant r_0 as in (39) (this corresponds simply to a choice of the origin of time in the far zone).

Now from the post-Minkowskian algorithm of Section 3, it is clear that the radiative moments U_L and V_L can be obtained to any post-Minkowskian order in principle, in the form of a non-linear series in the source or equivalently the canonical multipole moments M_L and S_L. The practical detail (worked out in [29]) is to determine the transformation between harmonic and radiative coordinates, generalizing (53) to any post-Minkowskian order. The structure of e.g. the mass-type radiative moment is

$$U_L = M_L^{(l)} + \sum_{n=2}^{+\infty} \frac{G^{n-1}}{c^{3(n-1)+2k}} X_{nL} \ . \tag{5.3}$$

The first term comes from the fact that the radiative moment reduces at the linearized approximation to the (lth time derivative of the) source or canonical moment. The second term represents the series of non-linear corrections, each of them is given by a certain X_{nL} which is a n-linear functional of derivatives of multipole moments M_L or S_L. Furthermore we know from e.g. (48) and (50) that each new non-linear iteration (which always involves a retarded integral) brings *a priori* a new "hereditary" integration with respect to the previous approximation. So we expect that X_{nL} is of the form ($U \equiv T - R/c$)

$$X_{nL}(U) = \sum \int_{-\infty}^{U} du_1 \dots \int_{-\infty}^{U} du_n \mathcal{Z}_n(U, u_1, \dots, u_n) M_{L_1}^{(a_1)}(u_1) \dots S_{L_n}^{(a_n)}(u_n) \tag{5.4}$$

where $\mathcal{Z}_n$ denotes a certain kernel depending on time variables $U, u_1, \dots, u_n$, and where the sum refers to all possibilities of coupling together the n moments. [See (56) below for examples of kernels $\mathcal{Z}_2$ and $\mathcal{Z}_3$.] A useful information is obtained from imposing that $\mathcal{Z}_n$ be dimensionless; this yields the powers of G and $1/c$ in front of each non-linear term in (54), where k is the number of contractions among the indices present on the n moments (the current moments carrying their associated Levi-Civita symbol).

As an example of application of (54) let us suppose that one is interested in the 3PN or $1/c^6$ approximation. From (54) we have $3(n-1) + 2k = 6$, and we deduce that the only possibility is $n = 3$ (cubic non-linearity) and $k = 0$ (no contractions between the moments). From this we infer immediately that the only possible multipole interaction at that order is between two mass monopoles and a multipole, i.e. $M \times M \times M_L$. This corresponds to the "tails of tails" computed explicitly in (56) below.

B. The radiative quadrupole moment to 3PN order

To implement the formula (54) a tedious computation is to be done, following in details the post-Minkowskian algorithm of Section 4 augmented by explicit formulas such as (47)-(51), and changing the coordinates from harmonic to radiative according to the prescription in [29]. Here we present the result of the computation of the mass-type radiative quadrupole ($l = 2$) up to the 3PN order:

$$U_{ij}(U) = M_{ij}^{(2)}(U) + 2\frac{GM}{c^3} \int_0^{+\infty} d\tau M_{ij}^{(4)}(U - \tau) \left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{11}{12} \right]$$

$$
+\frac{G}{c^5}\Big\{-\frac{2}{7}\int_0^{+\infty} d\tau \left[M^{(3)}_{a<i}M^{(3)}_{j>a}\right](U-\tau) - \frac{2}{7}M^{(3)}_{a<i}M^{(2)}_{j>a}
$$
$$
-\frac{5}{7}M^{(4)}_{a<i}M^{(1)}_{j>a} + \frac{1}{7}M^{(5)}_{a<i}M_{j>a} + \frac{1}{3}\varepsilon_{ab<i}M^{(4)}_{j>a}S_b\Big\}
$$
$$
+2\left(\frac{GM}{c^3}\right)^2\int_0^{+\infty} d\tau M^{(5)}_{ij}(U-\tau)\left[\ln^2\left(\frac{c\tau}{2r_0}\right) + \frac{57}{70}\ln\left(\frac{c\tau}{2r_0}\right) + \frac{124627}{44100}\right]
$$
$$
+O\left(\frac{1}{c^7}\right) \ . \tag{5.5}
$$

Recall that in this formula the moment M_{ij} is the canonical moment which agrees with the source moment I_{ij} up to a 2.5PN term [see (46)], and that the source moment I_{ij} itself is given in terms of the pseudo-tensor of the source by (36a). See also the formulas (68)-(69) below for a more explicit expression of the source moment at the 2PN order [of course, to be consistent, one should use (56) conjointly with 3PN expressions of the source moments].

The "Newtonian" term in (56) corresponds to the quadrupole formalism. Next, there is a quadratic non-linear correction with multipole interaction $M \times M_{ij}$ representing the dominant effect of tails (scattering of linear waves off the space-time curvature generated by the mass M). This correction, computed in [69], is of order $1/c^3$ or 1.5PN and has the form of a hereditary integral with logarithmic kernel. The constant 11/12 depends on the coordinate system chosen to cover the source, here the harmonic coordinates; for instance the constant would be 17/12 in Schwarzschild-like coordinates [87,88]. The next correction, of order $1/c^5$ or 2.5PN, is constituted by quadratic interactions between two mass quadrupoles, and between a mass quadrupole and a constant current dipole [70]. This term contains a hereditary integral, of a type different from the tail integral, which is due to the gravitational radiation generated by the stress-energy distribution of linear waves [89–91,69]. Sometimes this integral is referred to as the non-linear memory integral because it corresponds to the contribution of gravitons in the so-called linear memory effect [92]. The non-linear memory integral can easily be found by using the effective stress-energy tensor of gravitational waves in place of the right side of (3); it follows also from rigorous studies of the field at future null infinity [93,94]. Finally, at 3PN order in (56) appears the dominant cubic non-linear correction, corresponding to the interaction $M \times M \times M_{ij}$ and associated with the tails of tails of gravitational waves [71].

C. Tail contributions in the total energy flux

Observable quantities at infinity are expressible in terms of the radiative mass and current multipole moments. For instance the total gravitational-wave power emitted in all spatial directions (total gravitational flux or "luminosity" $\mathcal{L}$) is given by the positive-definite multipolar series

$$
\mathcal{L} = \sum_{l=2}^{+\infty} \frac{G}{c^{2l+1}}\Big\{\frac{(l+1)(l+2)}{l(l-1)l!(2l+1)!!}U^{(1)}_L U^{(1)}_L
$$

$$+\frac{4l(l+2)}{c^2(l-1)(l+1)!(2l+1)!!}V_L^{(1)}V_L^{(1)}\Big\} \ . \tag{5.6}$$

In the case of inspiralling compact binaries (a most prominent source of gravitational waves) the rate of inspiral is fixed by the flux $\mathcal{L}$, which is therefore a crucial quantity to predict. Excitingly enough, we know that $\mathcal{L}$ should be predicted to 3PN order for detection and analysis of inspiralling binaries in future experiments [95,96].

To 3PN order we can use the relation (56) giving the 3PN radiative quadrupole moment. Here we concentrate our attention on tails and tails of tails. The dominant tail contribution at 1.5PN order yields correspondingly a contribution in the total flux (with $U = T - R/c$):

$$\mathcal{L}_{\rm tail} = \frac{4G^2M}{5c^8} I_{ij}^{(3)}(U) \int_0^{+\infty} d\tau I_{ij}^{(5)}(U-\tau) \left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{11}{12}\right] \ . \tag{5.7}$$

Since we are interested in the dominant tail we have replaced using (46) the canonical mass quadrupole by the source quadrupole. Similarly there are some tail contributions due to the mass octupole, current quadrupole and all higher-order multipoles, but these are correlatively of higher post-Newtonian order [see the factors $1/c$ in (57)]. It has been shown [68] that the work done by the dominant "hereditary" contribution in the radiation reaction force within the source – which arises at 4PN order in the equations of motion – agrees exactly with (58).

Next, because $\mathcal{L}$ is made of squares of (derivatives of) radiative moments, it contains a term with the square of the tail integral at 1.5PN. This term arises at the 3PN relative order and reads

$$\mathcal{L}_{({\rm tail})^2} = \frac{4G^3M^2}{5c^{11}} \left(\int_0^{+\infty} d\tau I_{ij}^{(5)}(U-\tau) \left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{11}{12}\right]\right)^2 \ . \tag{5.8}$$

Finally, there is also the direct 3PN contribution of tails of tails in (56):

$$\mathcal{L}_{\rm tail(tail)} = \frac{4G^3M^2}{5c^{11}} I_{ij}^{(3)}(U) \int_0^{+\infty} d\tau I_{ij}^{(6)}(U-\tau) \times \left[\ln^2\left(\frac{c\tau}{2r_0}\right) + \frac{57}{70}\ln\left(\frac{c\tau}{2r_0}\right) + \frac{124627}{44100}\right] \ . \tag{5.9}$$

By a control of all the hereditary integrals in $\mathcal{L}$ up to 3PN we have checked [71] that the terms (59)-(60) do exist. The two contributions (59) and (60) appear somewhat on the same footing – of course both should be taken into account in practical computations. Note that in a physical situation where the emission of radiation stops after a certain date, in the sense that the source multipole moments become constant after this date (assuming a consistent matter model which would do this at a given post-Newtonian order), the only contribution to $\mathcal{L}$ which survives after the end of emission is the 3PN tail-square contribution (59).

VI. POST-NEWTONIAN APPROXIMATION

In Sections 2 and 3 we have reasoned upon the formal post-Newtonian expansion $\overline{h}^{\mu\nu}$ of the near-zone field to obtain the source multipole moments as functionals of the post-Newtonian pseudo-tensor $\overline{\tau}^{\mu\nu}$. We have also considered in Sections 4 and 5 the formal expansion $c \to \infty$ of the radiation field when holding the multipole moments fixed. Clearly missing in this scheme is an *explicit* algorithm for the computation of $\overline{h}^{\mu\nu}$ in the near zone. No such algorithm (say, in the spirit of the post-Minkowskian algorithm in Section 4) is known presently, but a lot is known on the first few post-Newtonian iterations [20,21,44–58,30,24].

The main difficulty in setting up a post-Newtonian algorithm is the appearance at some post-Newtonian order of divergent Poisson-like integrals. This comes from the fact that the post-Newtonian expansion is actually a near-zone expansion [44], which is valid only in the region where $r = O(\lambda/c)$, and that such an expansion blows up when taking formally the limit $r \to +\infty$. For instance, Rendall [13] has shown that the post-Newtonian solution cannot be asymptotically flat starting at the 2PN or 3PN level, depending on the gauge. This is clear from the structure of the exterior near-zone expansion (15), which involves many positive powers of the radial distance r. Thus, one is not allowed in general to consider the limit $r \to +\infty$. In consequence, using the Poisson integral for solving a Poisson equation with non-compact-support source at a given post-Newtonian order is *a priori* meaningless. Indeed the Poisson integral not only extends over the near-zone but also over the regions at infinity. This means that the Poisson integral does not constitute the correct solution of the Poisson equation in this context. However, to the lowest post-Newtonian orders it works; for instance it was shown by Kerlick [50,51] and Caporali [52] that the post-Newtonian iteration (including the suggestion by Ehlers [48,49] of an improvement with respect to previous work [55]) is well-defined up to the 2.5PN order where radiation reaction terms appear, but that some divergent integrals show up at the 3PN order.

Another difficulty is that the post-Newtonian approximation is in a sense not self-supporting, because it necessitates information coming from outside its own domain of validity. Of course we have in mind the boundary conditions at infinity which determine the radiation reaction in the source's local equations of motion. Again, to the lowest post-Newtonian orders one can circumvent this difficulty by considering *retarded* integrals that are formally expanded when $c \to \infty$ as series of "instantaneous" Poisson-like integrals [55]. However, this procedure becomes incorrect at the 4PN order, not to mention the problem of divergencies, because the near-zone field (as well as the source's dynamics) ceases to be given by an instantaneous functional of the source parameters, due to the appearance of "tail-transported" hereditary integrals modifying the lowest-order radiation reaction damping [68,32].

Let us advocate here that the cure of the latter difficulty (and perhaps of all difficulties) is the matching equation (16). Indeed suppose that one knows a particular solution of the Poisson equation at some post-Newtonian order. This solution might be in the form of some "finite part" of a Poisson integral. The correct post-Newtonian solution will be the sum of this particular solution and of a homogeneous solution satisfying the Laplace equation, namely a harmonic solution, regular at the origin, which can always be written in the form $\sum A_L \hat{x}_L$, for some unknown constant tensors A_L. The homogeneous solution is associated with radiation reaction effects. Now the matching equation states that the multipole expansion of the post-Newtonian solution agrees with the near-zone expansion of the exterior field (which has been computed beforehand in Section 4). The multipole expansion of the known particular solution can be obtained by a standard method, and the multipole expansion of the homogeneous solution is simply itself, i.e. $\mathcal{M}(\sum A_L \hat{x}_L) = \sum A_L \hat{x}_L$. Therefore, we see that the matching equation determines in principle the homogeneous solution (i.e. all the unknown tensors A_L), and since the exterior field satisfies relevant boundary conditions at infinity, the A_L's should correspond to the radiation reaction on a truly isolated system. See [67,68,31,32] for implementation of this method to determine the radiation reaction force to 4PN order (1.5PN relative order).

A. The inner metric to 2.5PN order

Going to high post-Newtonian orders can become prohibitive because of the rapid proliferation of terms. Typically any allowed term (compatible dimension, correct index structure) does appear with a definite non-zero coefficient in front. However, high post-Newtonian orders can be manageable if one chooses some appropriate matter variables, and if one avoids expanding systematically the retardations due to the speed of propagation of gravity. Often it is sufficient, and clearer, to present a result in terms of matter variables still containing some c's, and perhaps also in terms of some convenient retarded potentials (being clear that any retardation going to an order higher than the prescribed post-Newtonian order of the calculation is irrelevant). See for instance (65) and (68)-(69) below. Anyway, only in a final stage, when a result to the prescribed order is in hands, should we introduce the more basic matter variables (e.g. the coordinate mass density) and perform all necessary retardations. Then of course one does not escape to a profusion of terms, but at least we have been able to carry the post-Newtonian iteration using some reasonably simple expressions.

The matter variables are chosen [30,24] in a way consistent with our earlier definitions (35), i.e.

$$\sigma \equiv \frac{T^{00}+T^{ii}}{c^2} \; ; \tag{6.1a}$$

$$\sigma_i \equiv \frac{T^{0i}}{c} \; ; \tag{6.1b}$$

$$\sigma_{ij} \equiv T^{ij} \; . \tag{6.1c}$$

To 2.5PN order one defines some *retarded* potentials V, V_i, $\hat{W}_{ij}$, $\hat{X}$ and $\hat{R}_i$, with V and V_i looking like some retarded versions of the Newtonian and gravitomagnetic potentials, and $\hat{W}_{ij}$ being associated with the matter and gravitational-field stresses:

$$V \equiv \Box_R^{-1} \{-4\pi G\sigma\} \ , \tag{6.2a}$$

$$V_i \equiv \Box_R^{-1} \{-4\pi G\sigma_i\} \ , \tag{6.2b}$$

$$\hat{W}_{ij} \equiv \Box_R^{-1} \{-4\pi G(\sigma_{ij} - \delta_{ij}\sigma_{kk}) - \partial_i V \partial_j V\} \ , \tag{6.2c}$$

$$\hat{R}_i \equiv \Box_R^{-1} \left\{-4\pi G(V\sigma_i - V_i\sigma) - 2\partial_k V \partial_i V_k - \frac{3}{2}\partial_t V \partial_i V\right\} \ , \tag{6.2d}$$

$$\hat{X} \equiv \Box_R^{-1}\Big\{-4\pi G V\sigma_{ii} + 2V_i\partial_t\partial_i V + V\partial_t^2 V + \frac{3}{2}(\partial_t V)^2 - 2\partial_i V_j \partial_j V_i + \hat{W}_{ij}\partial_{ij}^2 V\Big\} \ , \tag{6.2e}$$

where $\Box_R^{-1}$ denotes the retarded integral (11). All these potentials but V and V_i have a spatially non-compact support. The highest non-linearity entering them is cubic; it appears in the last term of $\hat{X}$.

Based on the latter potentials one can show [24,74] that the inner metric to order 2.5PN (in harmonic coordinates, $\partial_\nu(\sqrt{-g}g^{\mu\nu}) = 0$) takes the form

$$g_{00} = -1 + \frac{2}{c^2}V - \frac{2}{c^4}V^2 + \frac{8}{c^6}\left[\hat{X} + V_iV_i + \frac{V^3}{6}\right] + O\left(\frac{1}{c^8}\right) \ , \tag{6.3a}$$

$$g_{0i} = -\frac{4}{c^3}V_i - \frac{8}{c^5}\hat{R}_i + O\left(\frac{1}{c^7}\right) \ , \tag{6.3b}$$

$$g_{ij} = \delta_{ij}\left(1 + \frac{2}{c^2}V + \frac{2}{c^4}V^2\right) + \frac{4}{c^4}\hat{W}_{ij} + O\left(\frac{1}{c^6}\right) \ , \tag{6.3c}$$

(writing $\overline{g}_{\mu\nu}$ would be more consistent with the notation of Section 2). With this form, we believe, the computational problems encountered in applications are conveniently divided into the specific problems associated with the computation of the various potentials (62), which constitute in this approach some appropriate computational "blocks" (having of course no physical signification separately). By expanding all powers of $1/c$ present into the matter densities (61) and into the retardations of the potentials (62), we find that the metric (63) becomes extremely complicated, as it really is (see e.g. [46,47,50,51]).

Because of our use of retarded potentials, the metric (63) involves explicitly only even post-Newtonian terms (using the post-Newtonian terminology that even terms correspond to even powers of $1/c$ in the equations of motion). We have checked [24] that the *odd* post-Newtonian terms (responsible for radiation reaction), contained in (63) via the expansion of retardations, match, in the sense of the equation (16), to the exterior metric satisfying the no-incoming radiation condition (9).

The harmonic gauge condition implies some differential equations to be satisfied by the previous potentials. To 2.5PN order we find

$$\partial_t\left\{V + \frac{1}{c^2}\left[\frac{1}{2}\hat{W}_{ii} + 2V^2\right]\right\} + \partial_i\left\{V_i + \frac{2}{c^2}\left[\hat{R}_i + VV_i\right]\right\} = O\left(\frac{1}{c^4}\right) \ , \tag{6.4a}$$

$$\partial_t V_i + \partial_j\left\{\hat{W}_{ij} - \frac{1}{2}\delta_{ij}\hat{W}_{kk}\right\} = O\left(\frac{1}{c^2}\right) \ , \tag{6.4b}$$

where $\hat{W}_{ii} \equiv \delta_{ij}\hat{W}_{ij}$. These equations are in turn equivalent to the equation of continuity and the equation of motion for the matter system,

$$\partial_t \sigma + \partial_i \sigma_i = \frac{1}{c^2}\left(\partial_t \sigma_{ii} - \sigma \partial_t V\right) + O\left(\frac{1}{c^4}\right) , \tag{6.5a}$$

$$\partial_t \sigma_i + \partial_j \sigma_{ij} = \sigma \partial_i V + O\left(\frac{1}{c^2}\right) . \tag{6.5b}$$

Note that the precision is 1PN for the equation of continuity but only Newtonian for the equation of motion.

B. The mass-type source moment to 2.5PN order

From the 2.5PN metric (63) we obtain the pseudo-tensor $\overline{\tau}$ and the auxiliary quantities (35), that we replace into the formulas (36) to obtain the 2.5PN source multipole moments. Recall that the z-integration in the moments is to be carried out using the formula (27). Let us first see how this works at the 1PN order.

We need Σ to 1PN order and Σ_i to Newtonian order. The latter quantity reduces to the matter part, $\Sigma_i = \sigma_i + O(1/c^2)$, and the former one reads after a simple transformation

$$\Sigma = \sigma - \frac{1}{2\pi G c^2}\Delta(V^2) + O\left(\frac{1}{c^4}\right) . \tag{6.6}$$

The substitution into the moments I_L given by (36a) leads to

$$\begin{aligned} I_L = \mathrm{FP}_{B=0} \int d^3\mathbf{x}\ |\mathbf{x}/r_0|^B \Big\{ & \hat{x}_L \sigma - \frac{\hat{x}_L}{2\pi G c^2}\Delta(V^2) \\ & + \frac{|\mathbf{x}|^2 \hat{x}_L}{2c^2(2l+3)}\partial_t^2 \sigma - \frac{4(2l+1)\hat{x}_{iL}}{c^2(l+1)(2l+3)}\partial_t \sigma_i \Big\} + O\left(\frac{1}{c^4}\right) . \end{aligned} \tag{6.7}$$

The integrand is non-compact-supported because of the contribution of the second term, and accordingly we keep the regularization factor $|\mathbf{x}/r_0|^B$ and finite part operation. But let us operate by parts the second term, using the fact that $|\mathbf{x}|^B \hat{x}_L \Delta(V^2) - \Delta(|\mathbf{x}|^B \hat{x}_L) V^2 = \partial_i \{ |\mathbf{x}|^B \hat{x}_L \partial_i (V^2) - \partial_i (|\mathbf{x}|^B \hat{x}_L) V^2 \}$ is a pure divergence. When the real part of B is a large *negative* number, we see thanks to the Gauss theorem that the latter divergence will not contribute to the moment, therefore by the unicity of the analytic continuation it will always yield zero contribution. Thus, using $\Delta \hat{x}_L = 0$, we can replace $|\mathbf{x}|^B \hat{x}_L \Delta(V^2)$ in the second term of (67) by $\Delta(|\mathbf{x}|^B \hat{x}_L) V^2 = B(B+l+1)|\mathbf{x}|^{B-2}\hat{x}_L V^2$, and because of the explicit factor B we see that the second term can be non-zero only in the case where the factor B multiplies an integral owning a simple pole $\sim 1/B$ due to the integration bound $|\mathbf{x}| \to \infty$. Expressing V^2 (to Newtonian order) in terms of source points $\mathbf{z}_1$ and $\mathbf{z}_2$, we obtain the integral $\int d^3\mathbf{x}\ |\mathbf{x}|^{B-2}\hat{x}_L |\mathbf{x} - \mathbf{z}_1|^{-1} |\mathbf{x} - \mathbf{z}_2|^{-1}$. When $|\mathbf{x}| \to \infty$ each $|\mathbf{x} - \mathbf{z}_{1,2}|^{-1}$ can be expanded as a series of $\hat{n}_{L_{1,2}} |\mathbf{x}|^{-l_{1,2}-1}$; then performing the angular integration shows that the sum of "multipolarities" $l + l_1 + l_2$ is necessarily an even integer. When this is realized the remaining radial integral reads $\int d|\mathbf{x}|\ |\mathbf{x}|^{B+l-l_1-l_2-2}$ which develops a pole only when $l-l_1-l_2-2=-1$. But that is incompatible with the previous finding. Thus the second term in (67) is identically zero, and we end up simply with a compact-support expression on which we no longer need to implement the finite part,

$$I_L = \int d^3\mathbf{x} \Big\{ \hat{x}_L \sigma + \frac{|\mathbf{x}|^2 \hat{x}_L}{2c^2(2l+3)} \partial_t^2 \sigma - \frac{4(2l+1)\hat{x}_{iL}}{c^2(l+1)(2l+3)} \partial_t \sigma_i \Big\} + O\left(\frac{1}{c^4}\right). \tag{6.8}$$

This expression was first obtained in [30] using a different method valid at 1PN order. Here we have recovered the same expression from the formula (36a) valid to any post-Newtonian order [24,25].

Only starting at the 2PN order does the mass multipole moment have a non-compact support (so the finite part becomes crucial at this order). By a detailed computation in [24] we arrive at the following 2PN (or rather 2.5PN) expression:

$$\begin{aligned} I_L(t) = \mathrm{FP}_{B=0} \int d^3\mathbf{x}\ |\mathbf{x}/r_0|^B \Big\{ & \hat{x}_L \left[\sigma + \frac{4}{c^4}\sigma_{ii} V \right] + \frac{|\mathbf{x}|^2 \hat{x}_L}{2c^2(2\ell+3)} \partial_t^2 \sigma \\ & + \frac{|\mathbf{x}|^4 \hat{x}_L}{8c^4(2\ell+3)(2\ell+5)} \partial_t^4 \sigma - \frac{2(2\ell+1)|\mathbf{x}|^2 \hat{x}_{iL}}{c^4(\ell+1)(2\ell+3)(2\ell+5)} \partial_t^3 \sigma_i \\ & + \frac{2(2\ell+1)\hat{x}_{ijL}}{c^4(\ell+1)(\ell+2)(2\ell+5)} \partial_t^2 \left[\sigma_{ij} + \frac{1}{4\pi G} \partial_i V \partial_j V \right] \\ & + \frac{\hat{x}_L}{\pi G c^4} \left[-\hat{W}_{ij} \partial^2_{ij} V - 2V_i \partial_t \partial_i V + 2\partial_i V_j \partial_j V_i - \frac{3}{2}(\partial_t V)^2 - V \partial_t^2 V \right] \\ & - \frac{4(2\ell+1)\hat{x}_{iL}}{c^2(\ell+1)(2\ell+3)} \partial_t \left[\left(1 + \frac{4V}{c^2} \right) \sigma_i \right. \\ & \left. + \frac{1}{\pi G c^2} \left(\partial_k V [\partial_i V_k - \partial_k V_i] + \frac{3}{4} \partial_t V \partial_i V \right) \right] \Big\} + O\left(\frac{1}{c^6}\right) . \end{aligned} \tag{6.9}$$

Recall that the canonical moment M_L differs from the source moment I_L at precisely the 2.5PN order [see (46)].

VII. POINT-PARTICLES

So far the post-Newtonian formalism has been developed for *smooth* (i.e. C^∞) matter distributions. As such, the source multipole moments (36) become ill-defined in the presence of singularities. We now argue that the formalism is in fact also applicable to singular sources (notably point-particles described by Dirac measures) provided that we add to our other basic assumptions a certain method for removing the infinite self-field of point-masses. Our main motivation is the inspiralling compact binary – a system of two compact objects (neutron stars or black holes) which can be described with great precision by two point-particles moving on a circular orbit, and whose orbital phase evolution should be computed prior to gravitational-wave detection with relative 3PN precision [95,96].

For this application we restrict ourselves to two point-masses m_1 and m_2 (constant Schwarzschild masses). The trajectories are $\mathbf{y}_1(t)$ and $\mathbf{y}_2(t)$ and the coordinate velocities

$\mathbf{v}_{1,2} = d\mathbf{y}_{1,2}/dt$; we pose $v^\mu_{1,2} = (c, \mathbf{v}_{1,2})$. The symbol $1 \leftrightarrow 2$ means the same term but with the labels of the two particles exchanged. A model for the stress-energy tensor of point-masses (say, at 2PN order) is

$$T^{\mu\nu}_{\rm point-mass}(\mathbf{x}, t) = \mu_1(t) v_1^\mu(t) v_1^\nu(t) \delta[\mathbf{x} - \mathbf{y}_1(t)] + 1 \leftrightarrow 2 \; ; \tag{7.1a}$$

$$\mu_1(t) \equiv \frac{m_1}{\sqrt{(g g_{\rho\sigma})_1 \frac{v_1^\rho v_1^\sigma}{c^2}}} \; , \tag{7.1b}$$

where δ denotes the three-dimensional Dirac measure, and $g_{\mu\nu}$ the metric coefficients in harmonic coordinates ($g \equiv \det g_{\mu\nu}$). The notation $(g g_{\mu\nu})_1$ means the value at the location of particle 1. However, due to the presence of the Dirac measure at particles 1 and 2, the metric coefficients will be singular at 1 and 2. Therefore, we must supplement the model (70) by a method of "regularization" able to give a sense to the ill-defined limit at 1 or 2. *A priori* the choice of one or another regularization constitutes a fully-qualified element of the model of point-particles. In the following we systematically employ the Hadamard regularization, based on the Hadamard "partie finie" of a divergent integral [97,98].

Let us discuss an example. The "Newtonian" potential U, defined by $U = \Delta^{-1}(-4\pi G\sigma)$, where σ is given by (61a) [we have $V = U + O(1/c^2)$], follows from (70a) as

$$U = \frac{G\mu_1}{r_1}\left[1 + \frac{v_1^2}{c^2}\right] + 1 \leftrightarrow 2 \; , \tag{7.2}$$

where $r_1 = |\mathbf{x} - \mathbf{y}_1|$. To Newtonian order $U = Gm_1/r_1 + O(1/c^2) + 1 \leftrightarrow 2$. We compute U at the 1PN order: from (70b) we deduce at this order $\mu_1/m_1 = 1 - (U)_1/c^2 + v_1^2/2c^2 + O(1/c^4)$, which involves U itself taken at point 1, but of course this does not make sense because U is singular at 1 and 2. Now, after applying the Hadamard regularization (described below), we obtain unambiguously the standard Newtonian result $(U)_1 = Gm_2/r_{12} + O(1/c^2)$, where $r_{12} = |\mathbf{y}_1 - \mathbf{y}_2|$, that we insert back into μ_1. So, U at 1PN, and its regularized value at 1, read

$$U = \frac{Gm_1}{r_1}\left(1 + \frac{1}{c^2}\left[-\frac{Gm_2}{r_{12}} + \frac{3}{2}v_1^2\right]\right) + O\left(\frac{1}{c^4}\right) + 1 \leftrightarrow 2 \; , \tag{7.3a}$$

$$(U)_1 = \frac{Gm_2}{r_{12}}\left(1 + \frac{1}{c^2}\left[-\frac{Gm_1}{r_{12}} + \frac{3}{2}v_2^2\right]\right) + O\left(\frac{1}{c^4}\right) \; . \tag{7.3b}$$

A. Hadamard partie finie regularization

We consider the class of functions of the field point $\mathbf{x}$ which are smooth on $I\!\!R^3$ except at the location of the two source points $\mathbf{y}_{1,2}$, around which the functions admit some power-like expansions in the radial distance $r_1 = |\mathbf{x} - \mathbf{y}_1|$, with fixed spatial direction $\mathbf{n}_1 = (\mathbf{x} - \mathbf{y}_1)/r_1$ (and idem for 2). Thus, for any $F(\mathbf{x})$ in this class, we have

$$F = \sum_a r_1^a f_{1(a)}(\mathbf{n}_1) \qquad (\text{when } r_1 \to 0) \; ; \tag{7.4a}$$

$$F = \sum_a r_2^a f_{2(a)}(\mathbf{n}_2) \qquad (\text{when } r_2 \to 0) \; , \tag{7.4b}$$

where the summation index a ranges over values in $\mathbb{Z}$ bounded from below, $a \geq -a_0$ (we do not need to be more specific), and where the coefficients of the various powers of $r_{1,2}$ depend on the spatial directions $\mathbf{n}_{1,2}$. In (73) we do not write the remainders for the expansions because we don't need them; simply, we regard the expansions (73) as listings of the various coefficients $f_{1(a)}$ and $f_{2(a)}$. We assume also that the functions F in this class decrease sufficiently rapidly when $|\mathbf{x}| \to \infty$, so that all integrals we consider are convergent at infinity.

The integral $\int d^3\mathbf{x} F$ is in general divergent because of the singular behaviour of F near $\mathbf{y}_{1,2}$, but we can compute its partie finie (Pf) in the sense of Hadamard [97,98]. Let us consider two volumes surrounding the two singularities, of the form $r_1 \leq s\rho_1(\mathbf{n}_1)$ (and similarly for 2), where s measures the size of the volume and ρ_1 gives its shape as a function of the direction $\mathbf{n}_1$ ($\rho_1 = 1$ in the case of a spherical ball). Using (73) it is easy to determine the expansion when $s \to 0$ of the integral extending on $\mathbb{R}^3$ deprived from the two previous volumes, and then to subtract from the integral all the divergent terms when $s \to 0$ in the latter expansion. The Hadamard partie finie is defined to be the limit when $s \to 0$ of what remains. As it turns out, the result can be advantageously re-expressed in terms of an integral on $\mathbb{R}^3$ deprived from two *spherical* balls ($\rho_{1,2} = 1$), at the price of introducing two constants $s_{1,2}$ which depend on the shape of the two regularizing volumes originally considered. With full generality the Hadamard partie finie of the divergent integral reads

$$\begin{aligned} \mathrm{Pf} \int d^3\mathbf{x} \; F \equiv \lim_{s \to 0} \Bigg\{ & \int_{\substack{r_1 > s \\ r_2 > s}} d^3\mathbf{x} \; F \\ & + \sum_{a+3 \leq -1} \frac{s^{a+3}}{a+3} \int d\Omega_1 f_{1(a)} + \ln\left(\frac{s}{s_1}\right) \int d\Omega_1 f_{1(-3)} + 1 \leftrightarrow 2 \Bigg\} \end{aligned} \tag{7.5}$$

where s_1 is given by

$$\ln s_1 = \frac{\int d\Omega_1 f_{1(-3)} \ln \rho_1}{\int d\Omega_1 f_{1(-3)}} \; . \tag{7.6}$$

Because of the two arbitrary constants $s_{1,2}$ the Hadamard partie finie is ambiguous, and one could think *a priori* that there is no point about defining a divergent integral by means of an ambiguous expression. Actually the point is that we control the origin of these constants: they come from the coefficients of $1/r_{1,2}^3$ in the expansions of F, which generate logarithmic terms in the integral. As we shall see the constants $s_{1,2}$ do not appear in the post-Newtonian metric up to the 2.5PN order (they are expected to appear only at 3PN order).

We can also give a meaning to the value of the function F at the location of particle 1 for instance, by taking the average over all directions $\mathbf{n}_1$ of the coefficient of the zeroth power of r_1 in (73a), namely

$$(F)_1 \equiv \int \frac{d\Omega_1}{4\pi} f_{1(0)} \ . \tag{7.7}$$

We refer also to the definition (76) as the Hadamard partie finie (of the function F at 1) because this definition is closely related to the definition (74) of the Hadamard partie finie of a divergent integral. To see this, apply (74) to the case where the function F is actually a gradient, $F = \partial_i G$, where G satisfies (73) [it is then clear that F itself satisfies (73)]. We find

$$\mathrm{Pf} \int d^3\mathbf{x}\ \partial_i G = -4\pi (n_1^i r_1^2 G)_1 - 4\pi (n_2^i r_2^2 G)_2 \tag{7.8}$$

where in the right side the values at 1 and 2 are taken in the sense of the Hadamard partie finie (76). This nice connection between the Hadamard partie finie of a divergent integral and that of a singular function is clearly understood from applying the Gauss theorem on two surfaces $r_{1,2} = s$ surrounding the singularities (there is no dependence on the constants $s_{1,2}$).

B. Multipole moments of point-mass binaries

To compute the source moments (36) of two point-particles we insert (70) in place of the stress-energy tensor $T^{\mu\nu}$ of a continuous source, and we pick up the Hadamard partie finie [in the sense of (74)] of all integrals. This *ansatz* reads

$$(I_L)_{\mathrm{point-mass}} = \mathrm{Pf}\left\{ I_L[T^{\mu\nu}_{\mathrm{point-mass}}] \right\} \ ; \tag{7.9a}$$

$$(J_L)_{\mathrm{point-mass}} = \mathrm{Pf}\left\{ J_L[T^{\mu\nu}_{\mathrm{point-mass}}] \right\} \ . \tag{7.9b}$$

As we have seen in (69), the source multipole moments involve at high PN order many (non-compact-support) non-linear contributions which can be expressed in terms of retarded potentials such as V. The paradigm of such non-linear contributions is a term involving the quadratic product of two (derivatives of) potentials V, say $\partial V \partial V$, or, neglecting $O(1/c^2)$ corrections, $\partial U \partial U$. To Newtonian order U is given by $Gm_1/r_1 + Gm_2/r_2$ and it is easily checked that this paradigmatic term can be written as a certain derivative operator, say $\partial\partial$, acting on the elementary integral (assuming for simplicity $l = 2$)

$$Y_{ij}(\mathbf{y}_1, \mathbf{y}_2) \equiv -\frac{1}{2\pi} \mathrm{FP}_{B=0} \int d^3\mathbf{x}\ |\mathbf{x}/r_0|^B \frac{\hat{x}_{ij}}{r_1 r_2} \ . \tag{7.10}$$

We see that the integral would be divergent at infinity without the finite part operation. However, it is perfectly well-behaved near 1 and 2 where there is no need of a regularization. The integral (79) can be evaluated in various ways; the net result is [24,72]

$$Y_{ij} = \frac{r_{12}}{3} \left[y_1^{<ij>} + y_1^{<i} y_2^{j>} + y_2^{<ij>} \right] \tag{7.11}$$

where $< ij >\equiv \mathrm{STF}(ij)$. Starting at 3PN order we meet some elementary integrals which need the regularization at 1 or 2 in addition to involving the finite part at infinity. An example is

$$Z_{ij}(\mathbf{y}_1) \equiv -\frac{1}{2\pi}\mathrm{Pf}\Big\{\mathrm{FP}_{B=0}\int d^3\mathbf{x}\ |\mathbf{x}/r_0|^B \frac{\hat{x}_{ij}}{r_1^3}\Big\}\ . \tag{7.12}$$

To obtain this integral one splits it into a near-zone integral extending over the domain $r_1 < \mathcal{R}_1$ (say), and a far-zone integral extending over $\mathcal{R}_1 < r_1$. The Hadamard regularization at 1 applies only to the near-zone integral, while the finite part at $B = 0$ is needed only for the far-zone integral. The result, found to be independent of the radius $\mathcal{R}_1$, reads [77]

$$Z_{ij} = \Big[2\ln\left(\frac{s_1}{r_0}\right) + \frac{16}{15}\Big]\, y_1^{<ij>}\ . \tag{7.13}$$

In this case we find an explicit dependence on both the constants r_0 due to the finite part at infinity, and s_1 due to the Hadamard partie finie near 1 [see (74)]. However these constants do not enter the multipole moments before the 3PN order (collaboration with Iyer and Joguet [77]).

A long computation, done in [72], yields the mass-type quadrupole moment at the 2PN order fully reduced in the case of two point-masses moving on a circular orbit. The method is to start from (69) (issued from [24]) and to employ notably the elementary integral (79)-(80) (see also [72] for the treatment of a cubically non-linear term). An equivalent result has been obtained by Will and Wiseman using their formalism [75]. In a mass-centered frame the moment is of the form

$$I_{ij} = \mu\Big(A\,\hat{y}_{ij} + B\,\frac{\hat{v}_{ij}}{\omega^2}\Big) + O\left(\frac{1}{c^5}\right)\ , \tag{7.14}$$

where $y_i = y_1^i - y_2^i$ and $v_i = v_1^i - v_2^i$, where ω denotes the binary's Newtonian orbital frequency [$\omega^2 = Gm/r_{12}^3$ with $m = m_1 + m_2$], and where $\mu = m_1 m_2/m$ is the reduced mass. The point is to obtain the coefficients A and B developed to 2PN order in terms of the post-Newtonian parameter $\gamma = Gm/r_{12}c^2$, where we recall that r_{12} is the distance between the two particles in harmonic coordinates. Untill 2PN we find some definite polynomials in the mass ratio $\nu = \mu/m$ (such that $0 < \nu \leq 1/4$):

$$A = 1 + \gamma\Big[-\frac{1}{42} - \frac{13}{14}\nu\Big] + \gamma^2\Big[-\frac{461}{1512} - \frac{18395}{1512}\nu - \frac{241}{1512}\nu^2\Big]\ , \tag{7.15a}$$

$$B = \gamma\Big[\frac{11}{21} - \frac{11}{7}\nu\Big] + \gamma^2\Big[\frac{1607}{378} - \frac{1681}{378}\nu + \frac{229}{378}\nu^2\Big]\ . \tag{7.15b}$$

The 2PN mass quadrupole moment (83)-(84) is part of a program aiming at computing the orbital phase evolution of inspiralling compact binaries to high post-Newtonian order (see Section 7.4). First-order black-hole perturbations, valid in the test-mass limit $\nu \to 0$ for one body, have already achieved the very high 5.5PN order [87,99–101]. Recovering the result of black-hole perturbations in this limit constitutes an important check of the overall formalism. For the moment it passed the check to 2.5PN order [72,73]; this is quite satisfactory regarding the many differences between the present approach and the black-hole perturbation method.

C. Equations of motion of compact binaries

The equations of motion of two point-masses play a crucial role in accounting for the observed dynamics of the binary pulsar PSR1913+16 [1–3,22], and constitute an important part of the program concerning inspiralling compact binaries. The motivation for investigating rigorously the equations of motion came in part from the salubrious criticizing remarks of Jürgen Ehlers *et al* [7]. Four different approaches have succeeded in obtaining the equations of motion of point-mass binaries complete up to the 2.5PN order (dominant order of radiation reaction): the "post-Minkowskian" approach of Damour, Deruelle and colleagues [16–19]; the "Hamiltonian" approach of Schäfer and predecessors [102,103,57,58] ; the "extended-body" approach of Kopejkin *et al* [104,105]; and the "post-Newtonian" approach of Blanchet, Faye and Ponsot [74]. The four approaches yield mutually agreeing results.

The post-Newtonian approach [74] consists of (i) inserting the point-mass stress-energy tensor (70) into the 2.5PN metric in harmonic coordinates given by (63); (ii) curing systematically the self-field divergences of point-masses using the Hadamard regularization; and (iii) substituting the regularized metric into the standard geodesic equations. For convenience we write the geodesic equation of the particle 1 in the Newtonian-like form

$$\frac{d\mathcal{P}_1^i}{dt} = \mathcal{F}_1^i \tag{7.16}$$

where the (specific) linear momentum $\mathcal{P}_1^i$ and force $\mathcal{F}_1^i$ are given by

$$\mathcal{P}_1^i = c\left(\frac{v_1^\mu g_{i\mu}}{\sqrt{-g_{\rho\sigma}v_1^\rho v_1^\sigma}}\right)_1 \quad ; \qquad \mathcal{F}_1^i = \frac{c}{2}\left(\frac{v_1^\mu v_1^\nu \partial_i g_{\mu\nu}}{\sqrt{-g_{\rho\sigma}v_1^\rho v_1^\sigma}}\right)_1 \quad . \tag{7.17}$$

Crucial in this method, the quantities are evaluated at the location of particle 1 according to the rule (76). All the potentials (62) and their gradients are evaluated in a way similar to our computation of U in (72), and then inserted into (85)-(86). We "order-reduce" the result, i.e. we replace each acceleration, consistently with the approximation, by its equivalent in terms of the positions and velocities as given by the (lower-order) equations of motion. After simplication we find, in agreement with other methods,

$$\begin{aligned}\frac{dv_1^i}{dt} = &-\frac{Gm_2}{r_{12}^2}n_{12}^i + \frac{Gm_2}{r_{12}^2c^2}\Big\{v_{12}^i\left[4(n_{12}v_1) - 3(n_{12}v_2)\right]\\ &+ n_{12}^i\left[-v_1^2 - 2v_2^2 + 4(v_1v_2) + \frac{3}{2}(n_{12}v_2)^2 + 5\frac{Gm_1}{r_{12}} + 4\frac{Gm_2}{r_{12}}\right]\Big\}\end{aligned}$$

$$
\begin{aligned}
&+ \frac{Gm_2}{r_{12}^2c^4} n_{12}^i \Big\{ \big[-2v_2^4 + 4v_2^2(v_1v_2) - 2(v_1v_2)^2 \\
&+ \frac{3}{2} v_1^2 (n_{12}v_2)^2 + \frac{9}{2} v_2^2 (n_{12}v_2)^2 - 6(v_1v_2)(n_{12}v_2)^2 - \frac{15}{8}(n_{12}v_2)^4 \Big] \\
&+ \frac{Gm_1}{r_{12}} \left[-\frac{15}{4} v_1^2 + \frac{5}{4} v_2^2 - \frac{5}{2}(v_1v_2) \right. \\
&\left. + \frac{39}{2}(n_{12}v_1)^2 - 39(n_{12}v_1)(n_{12}v_2) + \frac{17}{2}(n_{12}v_2)^2 \right] \\
&+ \frac{Gm_2}{r_{12}} \left[4v_2^2 - 8(v_1v_2) + 2(n_{12}v_1)^2 - 4(n_{12}v_1)(n_{12}v_2) - 6(n_{12}v_2)^2 \right] \\
&+ \frac{G^2}{r_{12}^2} \left[-\frac{57}{4} m_1^2 - 9m_2^2 - \frac{69}{2} m_1 m_2 \right] \Big\} \\
&+ \frac{Gm_2}{r_{12}^2c^4} v_{12}^i \Big\{ v_1^2 (n_{12}v_2) + 4v_2^2 (n_{12}v_1) - 5v_2^2 (n_{12}v_2) - 4(v_1v_2)(n_{12}v_1) \\
&+ 4(v_1v_2)(n_{12}v_2) - 6(n_{12}v_1)(n_{12}v_2)^2 + \frac{9}{2}(n_{12}v_2)^3 \\
&+ \frac{Gm_1}{r_{12}} \left[-\frac{63}{4}(n_{12}v_1) + \frac{55}{4}(n_{12}v_2) \right] + \frac{Gm_2}{r_{12}} \left[-2(n_{12}v_1) - 2(n_{12}v_2) \right] \Big\} \\
&+ \frac{4G^2m_1m_2}{5c^5r_{12}^3} \Big\{ n_{12}^i (n_{12}v_{12}) \left[-6\frac{Gm_1}{r_{12}} + \frac{52}{3}\frac{Gm_2}{r_{12}} + 3v_{12}^2 \right] \\
&+ v_{12}^i \left[2\frac{Gm_1}{r_{12}} - 8\frac{Gm_2}{r_{12}} - v_{12}^2 \right] \Big\} + O\left(\frac{1}{c^6}\right) \ ,
\end{aligned}
\tag{7.18}
$$

[where $n_{12}^i = (y_1^i - y_2^i)/r_{12}$; $v_{12}^i = v_1^i - v_2^i$; and e.g. $(n_{12}v_1)$ denotes the Euclidean scalar product]. At the 1PN or $1/c^2$ level the equations were obtained before by Lorentz an Droste [20], and by Einstein, Infeld and Hoffmann [21]. The 2.5PN or $1/c^5$ term represents the radiation damping in harmonic coordinates [correct because the metric (63) we started with matches to the post-Minkowskian exterior field]. In the case of circular orbits, the equations simplify drastically:

$$
\frac{dv_{12}^i}{dt} = -\omega_{2\mathrm{PN}}^2 y_{12}^i - \frac{32G^3m^3\nu}{5c^5r_{12}^4} v_{12}^i + O\left(\frac{1}{c^6}\right) \ , \tag{7.19}
$$

where the orbital frequency $\omega_{2\mathrm{PN}}$ of the 2PN circular motion reads

$$
\omega_{2\mathrm{PN}}^2 = \frac{Gm}{r_{12}^3} \left[1 + (-3+\nu)\gamma + \left(6 + \frac{41}{4}\nu + \nu^2 \right) \gamma^2 \right] \tag{7.20}
$$

(the post-Newtonian parameter is $\gamma = Gm/c^2 r_{12}$; and $\nu = \mu/m$).

D. Gravitational waveforms of inspiralling compact binaries

The gravitational radiation field and associated energy flux are given by (52) and (57) in terms of time-derivatives of the radiative multipole moments, themselves related to the source multipole moments by formulas such as (56). Furthermore, at a given post-Newtonian order, the source moments admit some explicit though complicated expressions such as (68)-(69), which, when specialized to (non-spinning) point-mass circular binaries, yield e.g. (83)-(84).

Now, for insertion into the radiation field and energy flux, one must compute the *time-derivatives* of the binary moments, with appropriate order-reduction using the binary's equations of motion (87)-(89). This yields in particular the fully reduced (up to the prescribed post-Newtonian order) gravitational waveform of the binary, or more precisely the two independent "plus" and "cross" polarization states h_+ and $h_\times$. The result to 2PN order is written in the form

$$h_{+,\times} = \frac{2Gm\nu x}{c^2 R}\left\{H^{(0)}_{+,\times} + x^{1/2}H^{(1/2)}_{+,\times} + xH^{(1)}_{+,\times} + x^{3/2}H^{(3/2)}_{+,\times} + x^2 H^{(2)}_{+,\times}\right\} , \tag{7.21}$$

where, for convenience, we have introduced a post-Newtonian parameter which is directly related to the orbital frequency: $x = (Gm\omega_{2\mathrm{PN}}/c^3)^{2/3}$, where $\omega_{2\mathrm{PN}}$ is given for circular orbits by (89). The various post-Newtonian coefficients in (90) depend on the cosine and sine of the "inclination" angle between the detector's direction and the normal to the orbital plane ($c_i = \cos i$ and $s_i = \sin i$), and on the masses through the ratios $\nu = \mu/m$ and $\delta m/m$, where $\delta m = m_1 - m_2$. The result for the "plus" polarization (collaboration with Iyer, Will and Wiseman [106]) is

$$H^{(0)}_+ = -(1+c_i^2)\cos 2\psi , \tag{7.22a}$$

$$H^{(1/2)}_+ = -\frac{s_i}{8}\frac{\delta m}{m}\Big[(5+c_i^2)\cos\psi - 9(1+c_i^2)\cos 3\psi\Big] , \tag{7.22b}$$

$$\begin{aligned} H^{(1)}_+ &= \frac{1}{6}\Big[19 + 9c_i^2 - 2c_i^4 - \nu(19 - 11c_i^2 - 6c_i^4)\Big]\cos 2\psi \\ &\quad -\frac{4}{3}s_i^2(1+c_i^2)(1-3\nu)\cos 4\psi , \end{aligned} \tag{7.22c}$$

$$\begin{aligned} H^{(3/2)}_+ &= \frac{s_i}{192}\frac{\delta m}{m}\Big\{\Big[57 + 60c_i^2 - c_i^4 - 2\nu(49 - 12c_i^2 - c_i^4)\Big]\cos\psi \\ &\quad -\frac{27}{2}\Big[73 + 40c_i^2 - 9c_i^4 - 2\nu(25 - 8c_i^2 - 9c_i^4)\Big]\cos 3\psi \\ &\quad +\frac{625}{2}(1-2\nu)s_i^2(1+c_i^2)\cos 5\psi\Big\} - 2\pi(1+c_i^2)\cos 2\psi , \end{aligned} \tag{7.22d}$$

$$\begin{aligned} H^{(2)}_+ &= \frac{1}{120}\Big[22 + 396c_i^2 + 145c_i^4 - 5c_i^6 + \frac{5}{3}\nu(706 - 216c_i^2 - 251c_i^4 + 15c_i^6) \\ &\qquad -5\nu^2(98 - 108c_i^2 + 7c_i^4 + 5c_i^6)\Big]\cos 2\psi \\ &\quad +\frac{2}{15}s_i^2\Big[59 + 35c_i^2 - 8c_i^4 - \frac{5}{3}\nu(131 + 59c_i^2 - 24c_i^4) \\ &\qquad +5\nu^2(21 - 3c_i^2 - 8c_i^4)\Big]\cos 4\psi \\ &\quad -\frac{81}{40}(1 - 5\nu + 5\nu^2)s_i^4(1+c_i^2)\cos 6\psi \\ &\quad +\frac{s_i}{40}\frac{\delta m}{m}\Big\{\Big[11 + 7c_i^2 + 10(5+c_i^2)\ln 2\Big]\sin\psi - 5\pi(5+c_i^2)\cos\psi \end{aligned}$$

$$-27\Big[7-10\,\ln(3/2)\Big](1+c_i^2)\sin 3\psi + 135\pi(1+c_i^2)\cos 3\psi\Big\}\ . \tag{7.22e}$$

The "cross" polarization admits a similar expression (see [106]). Here, ψ denotes a particular phase variable, related to the actual binary's orbital phase ϕ and frequency $\omega \equiv \omega_{2PN}$ by

$$\psi = \phi - \frac{2Gm\omega}{c^3}\ln\left(\frac{\omega}{\omega_0}\right)\ ; \tag{7.23}$$

ϕ is the angle, oriented in the sense of the motion, between the vector separation of the two bodies and a fixed direction in the orbital plane (since the bodies are not spinning, the orbital motion takes place in a plane). In (92), ω_0 denotes some constant frequency, for instance the orbital frequency when the signal enters the detector's frequency bandwidth; see [106] for discussion.

The previous formulas give the waveform of point-mass binaries whenever the frequency and phase of the orbital motion take the values ω and ϕ. To get the waveform as a function of time, we must replace ω and ϕ by their explicit time evolutions $\omega(t)$ and $\phi(t)$. Actually, the frequency is the time-derivative of the phase: $\omega = d\phi/dt$. The evolution of the phase is entirely determined, for circular orbits, by the energy balance equation $dE/dt = -\mathcal{L}$ relating the binding energy E of the binary in the center of mass to the emitted energy flux $\mathcal{L}$. E is computed using the equations of motion (87), and $\mathcal{L}$ follows from (57) and application of the previous formalism [changing the radiative moments to the source moments, applying (83)-(84), etc...]; the net result for the 2.5PN orbital phase [72,75,73] is

$$\begin{aligned}\phi = \phi_0 - \frac{1}{\nu}\Big\{&\Theta^{5/8} + \left(\frac{3715}{8064}+\frac{55}{96}\nu\right)\Theta^{3/8} - \frac{3}{4}\pi\Theta^{1/4}\\ &+\left(\frac{9275495}{14450688}+\frac{284875}{258048}\nu+\frac{1855}{2048}\nu^2\right)\Theta^{1/8}\\ &+\left(-\frac{38645}{172032}-\frac{15}{2048}\nu\right)\pi\ln\Theta\Big\}\ ,\end{aligned} \tag{7.24}$$

where ϕ_0 is a constant phase (determined for instance when the frequency is ω_0), and Θ the convenient dimensionless time variable

$$\Theta = \frac{c^3\nu}{5Gm}(t_c - t)\ , \tag{7.25}$$

t_c being the instant of coalescence at which, formally, $\omega(t)$ tends to infinity (of course, the post-Newtonian method breaks down before the final coalescence). All the results are in agreement, in the limit $\nu \to 0$, with those of black-hole perturbation theory [87,99–101].

VIII. CONCLUSION

The formalism reviewed in this article permits investigating in principle all aspects of the problem of dynamics and gravitational-wave emission of a *slowly-moving* isolated system (with, say, $v/c \sim 0.3$ at most): the generation of waves, their propagation in vacuum, the back-reaction onto the system, the structure of the asymptotic field, and most importantly the relation between the far-field and the source parameters. Of course, the formalism is merely post-Newtonian and never "exact", but in applications to astrophysical objects such as inspiralling compact binaries this should be sufficient provided that the post-Newtonian approximation is carried to high order.

Furthermore, there are several places in the formalism where some results are valid formally to any order of approximation. For instance, the source multipole moments are related to the *infinite* formal post-Newtonian expansion of the pseudo-tensor [see (18) or (36)], and the post-Minkowskian iteration of the exterior field is performed to *any* non-linear order [see (43)]. In such a situation, where an infinite approximate series can be defined, there is the interesting question of its relation to a corresponding element in the exact theory. For the moment the only solid work concerns the post-Minkowskian approximation of the exterior vacuum field, which has been proved to be asymptotic [26]. Likewise it is plausible that the expressions of the source multipole moments could be valid in the case of exact solutions.

The most important part of the formalism where a general prescription for how to proceed at *any* approximate step is missing, is the post-Newtonian expansion for the field inside the isolated system. For instance, though the multipole moments are given in terms of the formal post-Newtonian expansion of the pseudo-tensor, no general algorithm for computing *explicitly* this post-Newtonian expansion is known. An interesting task would be to define such an algorithm, in a manner similar to the post-Minkowskian algorithm in Section 4. In the author's opinion, the post-Newtonian algorithm should be defined conjointly with the post-Minkowskian algorithm, and should rely on the matching equation (16), so as to convey into the post-Newtonian field the information about the exterior metric.

Note that even if a general method for implementing a complete approximation series is defined, this method may be unworkable in practical calculations, because not explicit enough. For instance the post-Minkowskian series (43) is defined in terms of "iterated" retarded integrals, but needs to be suplemented by some formulas, to be used in applications, for the retarded integral of a multipolar extended source. In this respect it would be desirable to develop the formulas generalizing (50)-(51) to any non-linear order. This should permit in particular the study of the general structure of tails, tails of tails, and so on.

For the moment the only application of the formalism concerns the radiation and motion of point-particle binaries. Of course it is important to keep the formalism as general as possible, and not to restrict oneself to a particular type of source, but this application to point-particles offers some interesting questions. Indeed, it seems that the post-Newtonian approximation used conjointly with a regularization *à la* Hadamard works well, and that one is getting closer and closer to an exact (numerical) solution corresponding to the dynamics and radiation of two black-holes. So, in which sense does the post-Newtonian solution (corresponding to point-masses without horizons) approach a true solution for black-holes? Does the adopted method of regularizing the self-field play a crucial role? Is it possible to define a regularization consistently with the post-Newtonian approximation to all orders?

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